

Semiclassical equivalence of Green–Schwarz and Pure–Spinor/Hybrid formulations of superstrings in $AdS_5 \times S^5$ and $AdS_2 \times S^2 \times T^6$

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Abstract

We demonstrate the equivalence between the worldsheet one-loop partition functions computed near classical string solutions in the Green–Schwarz and in the pure–spinor formulations of superstrings in $AdS_5 \times S^5$. While their bosonic sectors are the same in the conformal gauge, their fermionic sectors superficially appear to be very different (1st vs 2nd–derivative kinetic terms, presence vs absence of fermionic gauge symmetry). Still, we show that the quadratic fluctuation spectrum of sixteen fermionic modes of the pure–spinor formulation is the same as in the Green–Schwarz superstring and the contribution of the extra “massless” fermionic modes cancels against that of the pure–spinor ghosts. We also provide evidence for a similar semiclassical equivalence between the Green–Schwarz and the hybrid formulations of superstrings in $AdS_2 \times S^2 \times T^6$ by studying several particular examples of string solutions.

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Contents

1	Introduction	1
2	$AdS_5 \times S^5$ Superstring	2
2.1	Green–Schwarz formulation	3
2.2	Pure–spinor formulation	4
2.3	Relation between Green–Schwarz and pure–spinor formulations	6
2.3.1	String motion in AdS_5	6
2.3.2	Generic string motion	7
2.3.3	Degenerate cases	8
2.4	Examples of string solutions in $AdS_5 \times S^5$	9
2.4.1	Infinite string in $AdS_2 \subset AdS_5$	9
2.4.2	Long spinning string in $AdS_3 \subset AdS_5$	11
3	$AdS_2 \times S^2 \times T^6$ superstring	12
3.1	Fermionic part of the GS action	13
3.2	Fermionic part of the hybrid model action	14
3.3	Examples of equivalence	15
3.3.1	Expansion near BMN geodesic	15
3.3.2	String wrapping an $S^1 \subset T^6$	17
3.4	Expansion near S^2 and T^2 worldsheet instantons	18
3.4.1	Green–Schwarz action	18
3.4.2	Hybrid model action	20
4	Concluding remarks	22
	Appendix A. Notation and conventions	23
	Appendix B. Folded spinning string in $R \times S^2 \subset AdS_2 \times S^2 \times T^6$	23

1 Introduction

To describe superstring theory on $AdS \times M$ backgrounds with Ramond–Ramond fluxes one can use either the Green–Schwarz (GS) formulation [1, 2, 3], or the pure–spinor (PS) formulation [4], or, in some low-dimensional cases, a hybrid model [5]. The case of the $AdS_5 \times S^5$ superstring has been studied in the GS and in the pure–spinor formulation (see, e.g., [6, 7, 8, 9] for review). In the less supersymmetric backgrounds $AdS_3 \times S^3 \times T^4$ and $AdS_2 \times S^2 \times T^6$ the GS formulation (see [10, 11] and references therein) and hybrid models [5, 12, 13] have been used.

These formulations are not on an equal footing. The GS action has a clear physical origin (describing, e.g., the motion of a fundamental string soliton in a type II supergravity background) and can be defined in a reparametrization–invariant way (though for perturbative quantization it requires a choice of a bosonic vacuum that spontaneously breaks some global symmetries). The PS formulation is defined as a fermionic extension of the bosonic string in conformal gauge with second-derivative kinetic terms for the fermions and ghosts added to ensure the BRST symmetry of the resulting 2d conformal theory. While this construction is somewhat ad hoc and the origin of the BRST invariance remains to be understood, it has the advantage that its perturbative quantization does not require a choice of a bosonic vacuum and thus can be, in principle, performed without breaking the underlying global symmetries.

The general relation between the type II GS superstring and the PS (or hybrid) formulation in non–trivial backgrounds remains an open problem.¹ Assuming it exists, such a relation is likely to require non–trivial (non–local) field redefinitions. A way towards understanding how the

¹For heterotic strings and in flat space this relation is, in fact, understood much better (see e.g. [14, 15, 16, 17, 18]). For generic curved superbackgrounds a passage from the GS action to the pure–spinor one [19] was studied in [20, 21, 22].

equivalence could be established is to study the correspondence between the quantum partition functions in the two formulations computed in the semiclassical expansion, i.e. by expanding near a classical bosonic string solution.

Almost all of the previous studies of semiclassical strings in $AdS_5 \times S^5$ and similar backgrounds was done using the GS formulation (see, e.g., [23, 24, 25, 7]). This is not too surprising given that the structure of the fermionic part of the pure-spinor action is much more complicated than that of the GS one. While the quadratic fermionic term in the GS action is roughly $\theta D\theta\partial x$ (becoming $\theta\tilde{D}\theta$ after a choice of bosonic x -background with 16 of 32 θ 's decoupled due to kappa-symmetry) the corresponding term in the PS action is $D\theta D\theta$ which has twice as many derivatives and none of the 32 θ 's decouple *a priori*.

The study of the semiclassical expansion of the pure-spinor superstring was so far done in the BMN limit [26], by expanding near a rigid circular 2-spin string in S^3 [27] and, more recently, for particular string solutions in the $R_t \times S^2$ part of $AdS_5 \times S^5$ [28]. The results were suggesting the equivalence (at least to 1-loop order in the semiclassical expansion) with the corresponding GS partition function.

In this paper we generalise the previous work of [27, 28] and explicitly prove the equivalence between the GS and PS worldsheet 1-loop partition functions computed by expanding near a generic bosonic string solution in $AdS_5 \times S^5$. We show, in particular, that 16 fermionic modes of the pure-spinor formulation have the same spectrum as the GS fermions while all the other modes decouple and effectively cancel against the pure-spinor ghost contribution in the partition function.

We also argue for a similar correspondence between the GS and the hybrid formulation for the $AdS_2 \times S^2 \times T^6$ superstring. In this case we do not provide a general proof but demonstrate the equivalence between the two fermionic sectors in several special cases. In particular, we consider the background of worldsheet instantons wrapping non-trivial cycles of $AdS_2 \times S^2 \times T^6$.

We start in section 2 with a review of the quadratic fermionic terms in the GS and in the PS actions expanded near a bosonic string solution. We then prove the equivalence of their contributions to the one-loop partition function evaluated near a generic classical string solution and further illustrate this equivalence on the examples of two simple infinite string solutions.

In section 3 we turn to the $AdS_2 \times S^2 \times T^6$ case and discuss the equivalence of the one-loop partition functions of the GS model and the hybrid model for a general motion of the classical string in $AdS_2 \times S^2$ and also, for a few simple cases, when the string moves in T^6 .

Appendix A contains some notation and conventions. In Appendix B we present a comparison between the GS and the hybrid model quadratic fluctuation Lagrangians in the case of a folded spinning string in the $R_t \times S^2$ part of $AdS_2 \times S^2$.

2 $AdS_5 \times S^5$ Superstring

In what follows we will be interested in one-loop partition functions computed by expansion near classical solutions, i.e. in the spectra of quadratic fluctuation operators. Since the pure spinor description of the superstring is *a priori* based on the conformal gauge, we will use this gauge also in the Green-Schwarz formulation.

The bosonic part of the Lagrangian which is the same for the GS and PS formulations is given by (we set the string tension to one)

$$\mathcal{L}_{\text{bose}} = \frac{1}{2}(g_{mn}^{AdS_5} \partial_i x^m \partial^i x^n + g_{m'n'}^{S^5} \partial_i y^{m'} \partial^i y^{n'}), \quad (2.1)$$

where x^m are AdS_5 coordinates, $y^{m'}$ are S^5 coordinates and $\xi^i = (\tau, \sigma)$ parametrize the worldsheet. We shall use the AdS_5 metric in global coordinates

$$ds^2 = -\cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, (d\theta^2 + \cos^2 \theta d\phi_1^2 + \sin^2 \theta d\phi_2^2). \quad (2.2)$$

The non-zero components of the corresponding AdS_5 spin connection are

$$\begin{aligned}\omega^{01} &= -\sinh \rho \, dt, & \omega^{12} &= \cosh \rho \, d\theta, & \omega^{13} &= \cos \theta \cosh \rho \, d\phi_1, \\ \omega^{14} &= \sin \theta \cosh \rho \, d\phi_2, & \omega^{23} &= -\sin \theta \, d\phi_1, & \omega^{24} &= \cos \theta \, d\phi_2.\end{aligned}\quad (2.3)$$

The Virasoro constraints on the bosonic fields are

$$G_{\tau\tau} + G_{\sigma\sigma} = 0, \quad G_{\tau\sigma} = 0, \quad G_{ij} \equiv g_{ij}^{AdS_5} + g_{ij}^{S^5} = e_i^A e_j^B \eta_{AB}, \quad (2.4)$$

where e_i^A are the worldsheet pullbacks of the $AdS_5 \times S^5$ vielbeins $\partial_i X^M e_M^A(X)$ ($A = a, a'$ and $X^M = x^m, y^{m'}$). These constraints express the fact that in the conformal gauge the induced metric is conformally flat

$$G_{ij} = \phi(\xi) \eta_{ij}. \quad (2.5)$$

The bosonic equations of motion in the conformal gauge are

$$\eta^{ij} \nabla_i e_j^A = 0, \quad (2.6)$$

where ∇_i is the worldsheet pullback of the $AdS_5 \times S^5$ covariant derivative. Note that the absence of torsion in the $AdS_5 \times S^5$ connection implies

$$\nabla_{[i} e_{j]}^A = 0. \quad (2.7)$$

2.1 Green–Schwarz formulation

The quadratic fermionic part of the GS Lagrangian on $AdS_5 \times S^5$ in the conformal gauge is (for our notation and conventions regarding spinors and gamma–matrices see Appendix A)

$$\mathcal{L}_{\text{GS}} = i\Theta \left(\eta^{ij} - \varepsilon^{ij} \sigma_3 \right) e_i^A \Gamma_A \left(\nabla_j + \frac{i}{2} \Gamma_{01234} \sigma_2 \Gamma_B e_j^B \right) \Theta, \quad (2.8)$$

where the covariant derivative is given by $\nabla_i = \partial_i - \frac{1}{4} \omega_i^{AB} \Gamma_{AB}$. Θ^I ($I = 1, 3$) are two 16–component Majorana–Weyl spinors of the same chirality and σ_2^{IJ} and σ_3^{IJ} are Pauli matrices. This Lagrangian can be obtained from the complete \mathbb{Z}_4 –graded sigma–model action on the supercoset $\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$ [3].²

The presence of the RR F_5 flux supporting the $AdS_5 \times S^5$ space manifests itself in the Γ_{01234} –term in the GS Lagrangian. To simplify the notation, let us define

$$\gamma_* \equiv \Gamma_{01234}, \quad \gamma_*^2 = -1 \quad (2.9)$$

$$\phi_i \equiv e_i^A \Gamma_A. \quad (2.10)$$

Then the Lagrangian (2.8) takes the form

$$\mathcal{L}_{\text{GS}} = i\Theta \left(\eta^{ij} - \varepsilon^{ij} \sigma_3 \right) \phi_i \mathcal{D}_j \Theta, \quad (2.11)$$

where

$$\mathcal{D}_j = \nabla_j + \frac{i}{2} \gamma_* \sigma_2 \phi_j \quad (2.12)$$

is a generalized covariant derivative appearing also in the Killing–spinor equation.

In terms of the \mathbb{Z}_4 –graded fermions $\Theta^I = (\Theta^1, \Theta^3)$, i.e. $\Theta^1 = \frac{1}{2}(1 + \sigma^3)\Theta$ and $\Theta^3 = \frac{1}{2}(1 - \sigma^3)\Theta$, the Lagrangian (2.11) may be explicitly written as

$$\mathcal{L}_{\text{GS}} = -i\Theta^1 \phi_- \nabla_+ \Theta^1 - i\Theta^3 \phi_+ \nabla_- \Theta^3 - i\Theta^1 \phi_- \gamma_* \phi_+ \Theta^3, \quad (2.13)$$

²We use the indices $I = (1, 3)$ instead of $I = (1, 2)$ to indicate that $\Theta^I = (\Theta^1, \Theta^3)$ have, respectively, grading 1 and 3 with respect to the \mathbb{Z}_4 –automorphisms of the superisometry group $PSU(2, 2|4)$ of the $AdS_5 \times S^5$ superbackground.

where $\phi_{\pm} = \phi_{\tau} \pm \phi_{\sigma}$ and $\nabla_{\pm} = \nabla_{\tau} \pm \nabla_{\sigma}$.

The Virasoro constraints and the bosonic equations of motion satisfied by the classical string solutions, eqs. (2.4), (2.6) and (2.7), imply

$$(\phi_{+})^2 = 0 = (\phi_{-})^2, \quad \nabla_{-} \phi_{+} = 0 = \nabla_{+} \phi_{-}, \quad (2.14)$$

$$\phi_{+} \phi_{-} + \phi_{-} \phi_{+} = -4\phi(\xi), \quad (2.15)$$

where $\phi = \frac{1}{2} \eta^{ij} e_i^A e_j^B \eta_{AB} = -\frac{1}{2} e_+^A e_-^B \eta_{AB}$ is the conformal factor of the induced worldsheet metric (2.5).

Let us now introduce the two projectors³

$$P_{-} = -\frac{1}{4\phi(\xi)} \phi_{-} \phi_{+}, \quad P_{+} = -\frac{1}{4\phi(\xi)} \phi_{+} \phi_{-} \quad (2.16)$$

satisfying, in view of eqs. (2.14) and (2.15), the following relations

$$\begin{aligned} P_{+} + P_{-} &= 1, & P_{-} P_{-} &= P_{-}, & P_{+} P_{+} &= P_{+}, & P_{-} P_{+} &= 0, \\ \nabla_{-} P_{-} &= -\nabla_{-} P_{+}, & \nabla_{+} P_{-} &= -\nabla_{+} P_{+}, \\ (\nabla_{-} P_{-}) P_{+} &= P_{-} \nabla_{-} P_{+} = P_{+} \nabla_{+} P_{-} = (\nabla_{+} P_{+}) P_{-} = 0. \end{aligned} \quad (2.17)$$

From these it also follows that

$$\nabla_{-} P_{-} = (\nabla_{-} P_{-}) P_{-}, \quad \nabla_{+} P_{-} = P_{-} \nabla_{+} P_{-}, \quad \nabla_{-} P_{+} = P_{+} \nabla_{-} P_{+}, \quad \nabla_{+} P_{+} = (\nabla_{+} P_{+}) P_{+}. \quad (2.18)$$

Using the above relations, the GS Lagrangian (2.13) can be written as

$$\mathcal{L}_{\text{GS}} = -i\Theta_{+}^1 \phi_{-} \nabla_{+} \Theta_{+}^1 - i\Theta_{-}^3 \phi_{+} \nabla_{-} \Theta_{-}^3 - i\Theta_{+}^1 \phi_{-} \gamma_{*} \phi_{+} \Theta_{-}^3, \quad (2.19)$$

where

$$\Theta_{\pm}^{1,3} = P_{\pm} \Theta^{1,3}. \quad (2.20)$$

Thus half of Θ^1 and Θ^3 drop out of the Lagrangian (2.19). This is a consequence of the kappa-symmetry of the GS formulation. Indeed, the GS action is, in general, invariant under the off-shell kappa-symmetry under which not only the fermions but also the bosonic string coordinates and the worldsheet metric are transforming. However, when expanding near a classical bosonic solution satisfying the Virasoro conditions kappa-symmetry becomes equivalent simply to a shift of fermions implying a degeneracy of the fermionic kinetic term (i.e. decoupling of the gauge part of the fermions) with the corresponding projector being $\begin{pmatrix} P_{+} & 0 \\ 0 & P_{-} \end{pmatrix}$.

2.2 Pure-spinor formulation

The quadratic term of the Θ -fermion part of the pure-spinor Lagrangian on $AdS_5 \times S^5$ is given by⁴

$$\mathcal{L}_{\text{PS}} = i\Theta (\eta^{ij} - \varepsilon^{ij} \sigma_3) \phi_i \mathcal{D}_j \Theta - 2\eta^{ij} \mathcal{D}_i \Theta \gamma_{*} \sigma_2 \mathcal{D}_j \Theta, \quad (2.21)$$

where \mathcal{D}_j was defined in (2.12). Note that the first term of this Lagrangian is the same as the quadratic term in the GS Lagrangian (2.8) while the second term is of second order in derivatives of Θ . The presence of this second term breaks the kappa-symmetry that was present in the GS action. Instead, the PS action contains additional ghost fields and is required to be BRST invariant. The ghost sector consists of bosonic spinors λ , $\hat{\lambda}$, ω and $\hat{\omega}$ satisfying the pure spinor conditions

$$\lambda \Gamma^A \lambda = \hat{\lambda} \Gamma^A \hat{\lambda} = \omega \Gamma^A \omega = \hat{\omega} \Gamma^A \hat{\omega} = 0, \quad (2.22)$$

³We assume here that $\phi \neq 0$. This is not so for point-like string solutions which will be discussed further in what follows.

⁴For a detailed description of the PS action in the present context we refer the reader to [28].

which reduce the number of independent components of each of the ghost fields from 16 to 11. The quadratic part of the ghost Lagrangian is

$$\mathcal{L}_{\text{ghost}} = \omega \nabla_- \lambda + \hat{\omega} \nabla_+ \hat{\lambda}. \quad (2.23)$$

The Lagrangian (2.21) can be rewritten as follows

$$\mathcal{L}_{\text{PS}} = -i\varepsilon^{ij} \Theta \sigma_3 \ell_i \mathcal{D}_j \Theta - 2\eta^{ij} \nabla_i \Theta \gamma_* \sigma_2 \nabla_j \Theta + i\eta^{ij} \nabla_i \Theta \ell_j \Theta. \quad (2.24)$$

Another form, obtained by adding a total derivative, is

$$\mathcal{L}_{\text{PS}} = -2 \nabla_i \Theta \gamma_* \sigma_2 (\eta^{ij} + \varepsilon^{ij} \sigma_3) \mathcal{D}_j \Theta.$$

In terms of the \mathbb{Z}_4 -graded fermion components $\Theta = (\Theta^1, \Theta^3)$ this Lagrangian takes the following form

$$\mathcal{L}_{\text{PS}} = 4i \nabla_- \Theta^1 \gamma_* \nabla_+ \Theta^3 - i \Theta^1 \ell_+ \nabla_- \Theta^1 - i \Theta^3 \ell_- \nabla_+ \Theta^3. \quad (2.25)$$

To compare this Lagrangian with (2.19) we split $\Theta^{1,3}$ entering (2.25) as in (2.20) with the use of the projectors (2.16)–(2.18) and get

$$\begin{aligned} \mathcal{L}_{\text{PS}} &= 4i \nabla_- \Theta^1 (P_+ + P_-) \gamma_* (P_+ + P_-) \nabla_+ \Theta^3 - i \Theta_-^1 \ell_+ \nabla_- \Theta_-^1 - i \Theta_+^3 \ell_- \nabla_+ \Theta_+^3 \\ &= 4i \nabla_- \Theta_-^1 P_+ (\gamma_* - \gamma_* W_-^{-1} \gamma_*) P_+ \nabla_+ \Theta_+^3 - i \Theta_-^1 \ell_+ \nabla_- \Theta_-^1 - i \Theta_+^3 \ell_- \nabla_+ \Theta_+^3 \\ &\quad + 4i (\nabla_- \Theta_+^1 + \nabla_- \Theta_-^1 + \nabla_- \Theta_-^1 P_+ \gamma_* W_-^{-1}) W_- (\nabla_+ \Theta_-^3 + \nabla_+ \Theta_+^3 + W_-^{-1} \gamma_* P_+ \nabla_+ \Theta_+^3), \end{aligned} \quad (2.26)$$

where

$$W_- = P_- \gamma_* P_- \quad (2.27)$$

and W^{-1} is defined by

$$W_- W_-^{-1} = W_-^{-1} W_- = P_- . \quad (2.28)$$

Special cases in which W_-^{-1} does not exist will be discussed in Section 2.3.3 below. It should be pointed out that the quadratic derivative terms in the first and the second line of (2.26) contain the matrix γ_* which is sandwiched with different projectors P_+ and P_- , respectively. This is important for the proper separation of the terms containing Θ_+^1 and Θ_-^3 (second line) from the rest.

An important feature of this action is that (the derivatives of) Θ_+^1 and Θ_-^3 enter eq. (2.26) only linearly. They can thus be integrated out producing a “massless” determinant of $\nabla_+ \nabla_-$. We will then be left with the first line in (2.26) which should be compared with the GS action (2.19). Indeed, if we substitute in (2.26) $X_+^1 = \nabla_- \Theta_+^1 P_- \equiv \nabla_- \Theta_+^1$ and $X_-^3 = P_- \nabla_+ \Theta_-^3 \equiv \nabla_+ \Theta_-^3$, we find that the equations of motion of X_+^1 and X_-^3 are $X_+^1 = -(\nabla_- \Theta_-^1 + \nabla_- \Theta_-^1 P_+ \gamma_* W_-^{-1}) P_-$ and $X_-^3 = -P_- (\nabla_+ \Theta_+^3 + W_-^{-1} \gamma_* P_+ \nabla_+ \Theta_+^3)$. Then the second line of (2.26) vanishes, and we are left with the first line containing only Θ_-^1 , Θ_+^3 , $\nabla_- \Theta_-^1 P_+$ and $P_+ \nabla_+ \Theta_+^3$ (the position of the projectors P_- and P_+ in the above relations is important).

The integration over Θ_+^1 and Θ_-^3 in the second line of (2.26) requires extra care when the induced worldsheet geometry has a non-zero curvature. To have the equivalence with the GS formulation one should properly define the corresponding path integral measure (cf. [29]). We will discuss this issue in more detail on the example of an infinite string in AdS_2 in Section 2.4.1.

Let us now comment on the pure-spinor ghost sector (2.23). Performing the following transformations of the ghosts [28]

$$\lambda \rightarrow V \lambda, \quad \hat{\lambda} \rightarrow U \hat{\lambda}, \quad \omega \rightarrow \omega V^{-1}, \quad \hat{\omega} \rightarrow \hat{\omega} U^{-1} \quad (2.29)$$

such that⁵

$$V^{-1} \partial_- V = \frac{1}{4} \omega_-^{AB} \Gamma_{AB}, \quad U^{-1} \partial_+ U = \frac{1}{4} \omega_+^{AB} \Gamma_{AB}, \quad (2.30)$$

⁵One should not confuse these transformations with local Lorentz rotations. Using the latter one would not be able to completely remove the non-trivial spin connection from the derivatives. Note that being effectively one dimensional, eqs. (2.30) can always be solved for V and U .

the kinetic terms (2.23) for the transformed pure spinors will contain the trivial partial derivatives ∂_{\pm} only:

$$\mathcal{L}_{\text{ghost}} = \omega \partial_- \lambda + \hat{\omega} \partial_+ \hat{\lambda}, \quad (2.31)$$

so that their contribution to the 1-loop partition function will be given simply by massless flat space Laplace determinants.

Note that analogous transformations of Θ^1 and Θ^3 can be used [28] to convert the covariant derivatives of the fermionic Lagrangians (2.25), (2.26) and (2.36) into simple partial derivatives

$$\Theta^1 \rightarrow V \Theta^1, \quad \Theta^3 \rightarrow U \Theta^3. \quad (2.32)$$

Such transformations may be useful for simplifying the fermionic Lagrangian when considering particular examples of string solutions, but in the generic case the analysis of the Lagrangian in the form (2.25) turns out to be technically simpler.

2.3 Relation between Green–Schwarz and pure–spinor formulations

Below we will show the equivalence between the 1-loop partition functions in the GS and PS formulations computed for generic string solutions in $AdS_5 \times S^5$. It is useful to start with a simpler case in which the string moves only in AdS_5 .

2.3.1 String motion in AdS_5

In the special case when only AdS_5 (or, by analytic continuation, only S^5) string coordinates are non-zero the projectors $\not{\epsilon}_{\pm}$ and P_{\pm} (anti)commute with $\gamma_* = \Gamma_{01234}$ and the Lagrangian (2.26) simplifies to⁶

$$\begin{aligned} \mathcal{L}_{\text{PS}} &= 4i \nabla_- \Theta_-^1 P_+ \gamma_* P_+ \nabla_+ \Theta_+^3 - i \Theta_-^1 \not{\epsilon}_+ \nabla_- \Theta_-^1 - i \Theta_+^3 \not{\epsilon}_- \nabla_+ \Theta_+^3 \\ &\quad + 4i (\nabla_- \Theta_+^1 + \nabla_+ \Theta_-^1) P_- \gamma_* P_- (\nabla_+ \Theta_-^3 + \nabla_- \Theta_+^3) \\ &= -i \phi^{-1} \nabla_- \Theta_-^1 \not{\epsilon}_+ \gamma_* \not{\epsilon}_- \nabla_+ \Theta_+^3 - i \Theta_-^1 \not{\epsilon}_+ \nabla_- \Theta_-^1 - i \Theta_+^3 \not{\epsilon}_- \nabla_+ \Theta_+^3 \\ &\quad - i \phi^{-1} (\nabla_- \Theta_+^1 + \nabla_+ \Theta_-^1) \not{\epsilon}_- \gamma_* \not{\epsilon}_+ (\nabla_+ \Theta_-^3 + \nabla_- \Theta_+^3). \end{aligned} \quad (2.33)$$

Upon integrating out Θ_+^1 and Θ_-^3 , we are left with the effective Lagrangian given by the first line of (2.33); we will denote it as \mathcal{L}_1 . To compare it with the GS Lagrangian (2.19), let us rewrite \mathcal{L}_1 in the following first order form by introducing the Lagrange multiplier spinors Ψ^1 and Ψ^3 (i.e. integrating out Ψ^1 and Ψ^3 leads back to the first line of (2.33))

$$\begin{aligned} \mathcal{L}_1 &= -i \Psi_+^3 \not{\epsilon}_- \gamma_* \not{\epsilon}_+ \Psi_-^1 + 2i \Psi_-^1 \not{\epsilon}_+ \nabla_- \Theta_-^1 + 2i \Psi_+^3 \not{\epsilon}_- \nabla_+ \Theta_+^3 \\ &\quad - i \Theta_-^1 \not{\epsilon}_+ \nabla_- \Theta_-^1 - i \Theta_+^3 \not{\epsilon}_- \nabla_+ \Theta_+^3 \\ &= i \Psi_-^1 \not{\epsilon}_+ \nabla_- \Psi_-^1 + i \Psi_+^3 \not{\epsilon}_- \nabla_+ \Psi_+^3 + i \Psi_-^1 \not{\epsilon}_+ \gamma_* \not{\epsilon}_- \Psi_+^3 \\ &\quad - i \tilde{\Theta}_-^1 \not{\epsilon}_+ \nabla_- \tilde{\Theta}_-^1 - i \tilde{\Theta}_+^3 \not{\epsilon}_- \nabla_+ \tilde{\Theta}_+^3, \end{aligned} \quad (2.34)$$

where we introduced

$$\tilde{\Theta}_-^1 = \Theta_-^1 - \Psi_-^1, \quad \tilde{\Theta}_+^3 = \Theta_+^3 - \Psi_+^3. \quad (2.35)$$

From (2.34) we conclude that the action for sixteen independent fermions Ψ_-^1 and Ψ_+^3 is the same as for the GS fermions in (2.19),⁷ while the sixteen $\tilde{\Theta}_-^1$ and $\tilde{\Theta}_+^3$ modes decouple and contribute just “massless” determinants to the partition function.

⁶The sign of the first term can be changed by changing the sign of Θ^1 . Here ϕ is the conformal factor from (2.15).

⁷Note that in the corresponding GS action the ∇_+ derivative acts on Θ^1 and ∇_- acts on Θ^3 , i.e. relating the GS and the PS actions requires exchanging Θ^1 with Θ^3 [28]. As we have already mentioned, the fermionic part of the PS action can be formally obtained (see eq. (2.24)) by adding to the GS action an extra second-derivative term for the fermions. It then happens (as we shall also see on some examples) that the resulting model can be related to the GS superstring which has the opposite sign of the Wess–Zumino term compared to the original one we started with. This change of the WZ term sign is equivalent to interchanging Θ^1 and Θ^3 .

We thus conclude that when a classical string worldsheet is embedded only in AdS_5 (or only in S^5) the fermionic sector of the PS action produces the same one-loop contribution as the GS fermions up to additional “massless” determinants. The latter should be canceled by the pure-spinor ghost contributions as required by a consistent count of degrees of freedom (and as a necessary requirement for having a consistent flat-space limit).

2.3.2 Generic string motion

In the case of a generic classical string motion in $AdS_5 \times S^5$ the first line of the Lagrangian (2.26) can also be put into first order form analogous to (2.34)

$$\begin{aligned}\mathcal{L}_1 &= i\Psi_+^3 M^{-1}\Psi_-^1 + 2i\Psi_-^1 \not\!e_+ \nabla_- \Theta_-^1 + 2i\Psi_+^3 \not\!e_- \nabla_+ \Theta_+^3 - i\Theta_-^1 \not\!e_+ \nabla_- \Theta_-^1 - i\Theta_+^3 \not\!e_- \nabla_+ \Theta_+^3 \\ &= i\Psi_-^1 \not\!e_+ \nabla_- \Psi_-^1 + i\Psi_+^3 \not\!e_- \nabla_+ \Psi_+^3 + i\Psi_+^3 M^{-1}\Psi_-^1 - i\tilde{\Theta}_-^1 \not\!e_+ \nabla_- \tilde{\Theta}_-^1 - i\tilde{\Theta}_+^3 \not\!e_- \nabla_+ \tilde{\Theta}_+^3, \quad (2.36)\end{aligned}$$

where

$$M = \frac{1}{4\phi} \not\!e_- (\gamma_* - \gamma_* W_-^{-1} \gamma_*) \not\!e_+, \quad MM^{-1} = \not\!e_- \not\!e_+. \quad (2.37)$$

To establish the relation with the GS fermionic Lagrangian (2.19) it remains to show that the matrix M^{-1} coincides with $\not\!e_- \gamma_* \not\!e_+$. For a class of solutions describing strings moving in the $R_t \times S^2$ part of $AdS_5 \times S^5$ this was demonstrated in [28]. We shall now prove that this is so for any generic classical string solution.

We should first find W_-^{-1} of eq. (2.28). To this end let us analyse the structure of the matrix $W_- = P_- \gamma_* P_-$. We split $\not\!e_\pm$ as follows

$$\not\!e_\pm = \not\!e_\pm + \not\!s_\pm, \quad \not\!e_\pm \equiv e_\pm^a(x) \Gamma_a, \quad \not\!s_\pm \equiv e_\pm^{a'}(y) \Gamma_{a'} \quad (2.38)$$

where $\not\!e_\pm$ and $\not\!s_\pm$ are, respectively, the pullbacks of the gamma-contracted AdS_5 and S^5 vielbeins satisfying

$$\begin{aligned}\{\not\!e_\pm, \not\!s_\pm\} &= 0, \quad [\not\!e_\pm, \gamma_*] = 0, \quad \{\not\!s_\pm, \gamma_*\} = 0, \\ \not\!e_+ \not\!e_+ &= -\not\!s_+ \not\!s_+, \quad \not\!e_- \not\!e_- = -\not\!s_- \not\!s_-.\end{aligned} \quad (2.39)$$

The relations in the last line are due to the Virasoro constraints (2.14). Using these relations we find that

$$\begin{aligned}W_- &= P_- \gamma_* P_- = \frac{1}{16\phi^2} \not\!e_- \not\!e_+ \gamma_* \not\!e_- \not\!e_+ = \frac{1}{16\phi^2} e_- \gamma_* (e_+ - s_+) \not\!e_- \not\!e_+ \\ &= -\frac{1}{16\phi^2} \not\!e_- \gamma_* (e_+ - s_+) \not\!R = -\frac{1}{16\phi^2} \not\!e_- \not\!e_+ \gamma_* \not\!R \\ &= \frac{1}{4\phi} P_- \gamma_* \not\!R = \frac{1}{4\phi} \not\!R \gamma_* P_-,\end{aligned} \quad (2.40)$$

where

$$\not\!R = \not\!e_+ \not\!e_- - \not\!e_- \not\!e_+ + \not\!s_+ \not\!s_- - \not\!s_- \not\!s_+, \quad [\not\!R, \gamma_*] = 0. \quad (2.41)$$

Note that $\not\!R_{AdS_5} = \not\!e_+ \not\!e_- - \not\!e_- \not\!e_+$ and $\not\!R_{S^5} = -(\not\!s_+ \not\!s_- - \not\!s_- \not\!s_+)$ are the worldsheet pullbacks of the AdS_5 and S^5 curvature.

We also find that

$$\not\!e_+ \not\!e_- \gamma_* \not\!e_+ = \not\!R \gamma_* \not\!e_+. \quad (2.42)$$

For a generic classical string solution the matrix $\not\!R$ is invertible (special cases for which this is not true are discussed in the next section). Then

$$W_-^{-1} = -4\phi \not\!R^{-1} \gamma_* P_- = -4\phi P_- \gamma_* \not\!R^{-1}. \quad (2.43)$$

Substituting eq. (2.43) into (2.37) we get

$$M = \frac{1}{4\phi} \not\!e_- (\gamma_* - \gamma_* W_-^{-1} \gamma_*) \not\!e_+ = \frac{1}{4\phi} (\not\!e_- \gamma_* \not\!e_+ - 4\phi \not\!e_- \gamma_* P_- \not\!R^{-1} \not\!e_+). \quad (2.44)$$

Its ‘inverse’ matrix is $M^{-1} = \not\epsilon_- \gamma_* \not\epsilon_+$. Indeed, using eq. (2.42), one checks that

$$\begin{aligned} M \not\epsilon_- \gamma_* \not\epsilon_+ &= \frac{1}{4\phi} (\not\epsilon_- \gamma_* \not\epsilon_+ - 4\phi \not\epsilon_- \gamma_* P_- \not{R}^{-1} \not\epsilon_+) \not\epsilon_- \gamma_* \not\epsilon_+ \\ &= -\not\epsilon_- \gamma_* P_+ \gamma_* \not\epsilon_+ - \not\epsilon_- \gamma_* P_- \not{R}^{-1} \not\epsilon_+ \not\epsilon_- \gamma_* \not\epsilon_+ = -\not\epsilon_- \gamma_* P_+ \gamma_* \not\epsilon_+ - \not\epsilon_- \gamma_* P_- \gamma_* \not\epsilon_+ = \not\epsilon_- \not\epsilon_+. \end{aligned} \quad (2.45)$$

We have thus proved that the “mass matrix” M^{-1} which appears in the first-order form of the PS fermionic Lagrangian (2.36) coincides with that of the GS Lagrangian (2.19). This shows that around any classical string solution with non-degenerate matrix \not{R} in (2.41) sixteen fermions of the pure-spinor formulation have the same fluctuation spectrum as the GS fermions while the rest of the PS fermions are effectively “massless”.

To summarise, the correspondence between the one-loop partition functions in the GS and PS formulations can be described as follows. The contribution of the 10 bosonic fluctuation modes is of course the same in the two cases. In the GS formulation the 8 (pairs of) physical fermionic modes contribute the determinant of a Dirac-like operator and there is also a trivial (flat-space) conformal ghost determinant⁸ $(\det \partial^2)^2$. In the PS formulation the above analysis implies that 8 (pairs of) fermionic modes contribute the same determinant as the GS fermions while the remaining 24 fermionic modes produce massless determinants $(\det \partial^2)^{16+8} = (\det \partial^2)^{24}$. The factor $(\det \partial^2)^{16}$ comes from the integration of Θ_+^1 and Θ_-^3 in the second line of (2.26) while $(\det \partial^2)^8$ originates from the massless modes in (2.36). In the PS formulation there are no conformal ghosts but there are 22 chiral and 22 anti-chiral pure spinor ghosts⁹ which contribute $(\det \partial^2)^{-22}$. In total, we are left with the “massless” determinant factor $(\det \partial^2)^2$ which is the same as the contribution of the conformal ghosts in the GS formulation.

We thus conclude that the one-loop partition functions in the two formulations do match near a generic classical string solution.

2.3.3 Degenerate cases

There are two special classes of solutions which are not covered by the above analysis:

(i) Solutions with degenerate 2d induced metric for which the conformal factor vanishes, $\phi = 0$. This happens for point-like (*e.g.* BMN) solutions. In these cases we cannot directly define the projection operators in (2.16), and the analysis should be done, for instance, by first performing a suitable re-scaling of fermions with ϕ in the corresponding non-degenerate case and only then taking the limit $\phi \rightarrow 0$. The BMN limit of the $AdS_5 \times S^5$ superstring in the pure spinor formulation was studied in [26] and was shown to be equivalent to that of the GS superstring. In Section 3.3.1 we will show that a similar equivalence holds also between the GS and the hybrid formulation of the $AdS_2 \times S^2 \times T^6$ superstring with a BMN geodesic running along S^2 and T^6 .

(ii) Solutions for which \not{R} in (2.41) is not invertible. Note that the fact that $\not{R}(\not{R}_{AdS_5} + \not{R}_{S^5}) \propto \eta^{ij}(g_{ij}^{AdS_5} - g_{ij}^{S^5})$ means that \not{R} is invertible unless $\eta^{ij}g_{ij}^{AdS_5} = \eta^{ij}g_{ij}^{S^5} = \phi$. Whether \not{R} is invertible for solutions with $\eta^{ij}g_{ij}^{AdS_5} = \eta^{ij}g_{ij}^{S^5} = \phi$ should be checked case by case.

An example of a rigid rotating string for which \not{R} is not invertible is the $J \rightarrow 0$ limit of the circular string solution of [30] (see eqs. (3.12)-(3.13) in [31]) for which $\not{R} = \not{R}_{AdS_5} = \not{R}_{S^5} = 0$. As this is a limit of a solution for which \not{R} is invertible, we expect that the PS and GS formulations should still agree also in this limit. Let us now show this explicitly. In this case

$$\not\epsilon_\pm = \not\epsilon_\tau \pm \not\epsilon_\sigma \Rightarrow \not\epsilon_+ \gamma_* = \gamma_* \not\epsilon_- \Rightarrow \not\epsilon_+ \gamma_* \not\epsilon_- = \not\epsilon_- \gamma_* \not\epsilon_+ = P_\pm \gamma_* P_\pm = 0. \quad (2.46)$$

Since here the induced metric is flat we have $\nabla_\pm = \partial_\pm$ and, due to (2.14) and (2.46),

$$\partial_+ \not\epsilon_+ = -\partial_+(\gamma_* \not\epsilon_- \gamma_*) = 0, \quad \partial_- \not\epsilon_- = -\partial_-(\gamma_* \not\epsilon_+ \gamma_*) = 0 \Rightarrow \partial_\pm P_\pm = \partial_\mp P_\pm = 0. \quad (2.47)$$

⁸Here we are assuming that we use flat rather than an induced 2d metric to define the determinants in the conformal gauge, see [23] for a related discussion.

⁹To integrate the pure-spinor ghosts contributing into the path integral with the Lagrangian (2.23), one should take into account the constraints (2.22). At the expense of $D = 10$ covariance, each of the constraints can be solved explicitly in terms of eleven independent parameters.

As a result, the GS Lagrangian takes the simple “massless” form

$$\mathcal{L}_{\text{GS}} = -i\Theta_+^1 \not{\partial}_- \Theta_+^1 - i\Theta_-^3 \not{\partial}_+ \Theta_-^3 , \quad (2.48)$$

while the PS Lagrangian becomes

$$\mathcal{L}_{\text{PS}} = 4i \not{\partial}_- \Theta_-^1 \gamma_* \not{\partial}_+ \Theta_-^3 + 4i \not{\partial}_- \Theta_+^1 \gamma_* \not{\partial}_+ \Theta_+^3 - i\Theta_-^1 \not{\partial}_+ \Theta_-^1 - i\Theta_+^3 \not{\partial}_+ \Theta_+^3 .$$

From the form of this Lagrangian in which the (derivatives of) Θ_+^1 and Θ_-^3 enter only linearly it is clear that the integration over the fermionic fields will only contribute to the partition function with “massless” determinants: One finds indeed a factor $\sim (\det \partial^2)^{8+24}$ times the contribution $(\det \partial^2)^{-22}$ of the pure-spinor ghosts, which is the same as in the GS case.

The matrix \not{R} may, in principle, be degenerate also in (certain limits of) more complicated cases like elliptic strings or non-rigid strings. Then the number of “massive” fermionic modes in both the Green-Schwarz and the pure-spinor formulation will be determined by the rank of the matrix (2.42), i.e. $\not{\epsilon}_- \not{R} \gamma_* \not{\epsilon}_+$. The analysis will follow along the similar lines, by combining the consideration made in the above singular case and in the case of non-degenerate invertible part of $\not{\epsilon}_- \not{R} \gamma_* \not{\epsilon}_+$.

Below we will illustrate the general proof of the semiclassical equivalence of GS and PS formulations given above by explicitly verifying this equivalence for two simple examples of string solutions.

2.4 Examples of string solutions in $AdS_5 \times S^5$

We shall consider two simple limits of the folded spinning string moving in the AdS_3 part of AdS_5 [32, 24]. For generic values of parameters the corresponding string coordinates in the AdS_5 metric (2.2) are

$$t = \kappa\tau, \quad \phi_2 = \omega\tau, \quad \rho = \rho(\sigma), \quad \phi_1 = 0, \quad \theta = \frac{\pi}{2}, \quad (2.49)$$

where

$$\rho'^2 = \kappa^2 \cosh^2 \rho - \omega^2 \sinh^2 \rho, \quad \rho'' = (\kappa^2 - \omega^2) \sinh \rho \cosh \rho. \quad (2.50)$$

One special case [24] is when $\omega = 0$ with periodicity constraint in σ removed – one finds then an open-string solution that represents an infinite string in AdS_2 stretched all the way to the boundary with

$$t = \kappa\tau, \quad \tanh \frac{\rho}{2} = \tanh \frac{\kappa\sigma}{2}, \quad \rho' = \kappa \cosh \rho. \quad (2.51)$$

The corresponding induced metric is the AdS_2 one with the curvature $R^{(2)} = -2$.¹⁰

Another limiting case is that of an infinite-spin long folded string when $\kappa \rightarrow \infty$ so that

$$t = \phi_2 = \kappa\tau, \quad \rho = \kappa\sigma, \quad \phi_1 = 0, \quad \theta = \frac{\pi}{2}. \quad (2.52)$$

Here the string also reaches the boundary, but having a spin, it has to be embedded at least into AdS_3 .¹¹

2.4.1 Infinite string in $AdS_2 \subset AdS_5$

Choosing $\kappa = 1$ in (2.51) the corresponding AdS_2 metric and spin connection are

$$\begin{aligned} ds^2 &= -\cosh^2 \rho dt^2 + d\rho^2, & e^0 &= \cosh \rho dt, & e^1 &= d\rho, \\ \omega^{ab} &= -\sinh \rho dt \varepsilon^{ab}, & \nabla_\tau &= \partial_\tau + \frac{1}{2} \sinh \rho \Gamma_{01}, & \nabla_\sigma &= \partial_\sigma. \end{aligned} \quad (2.53)$$

¹⁰Written in Poincaré coordinates this Minkowski solution is equivalent to a worldsheet ending on an infinite straight line at the boundary. This is 1/2 BPS configuration; fluctuations near it were studied in [23].

¹¹After a Euclidean worldsheet continuation and conformal transformation the corresponding Poincaré patch solution is a null cusp surface [33].

Defining the $AdS_5 \times S^5$ fluctuation fields as

$$t = \tau + \frac{\tilde{t}(\tau, \sigma)}{\cosh \rho}, \quad \rho = \rho(\sigma) + \tilde{\rho}(\tau, \sigma), \quad \tilde{\theta}(\tau, \sigma), \quad \tilde{\phi}_{1,2}(\tau, \sigma), \quad \tilde{y}^{m'}(\tau, \sigma), \quad (2.54)$$

we can write the bosonic part of the quadratic fluctuation Lagrangian as¹²

$$\begin{aligned} \mathcal{L}_{\text{bose}} = & -\frac{1}{2}((\nabla \tilde{t})^2 + \cosh^2 \rho \tilde{t}^2) + \frac{1}{2}((\nabla \tilde{\rho})^2 + \cosh^2 \rho \tilde{\rho}^2) + \frac{1}{2}((\partial \tilde{\theta})^2 + 2 \cosh^2 \rho \tilde{\theta}^2) \\ & + \frac{1}{2}((\partial \tilde{\phi}_1)^2 + 2 \cosh^2 \rho \tilde{\phi}_1^2) + \frac{1}{2}((\partial \tilde{\phi}_2)^2 + 2 \cosh^2 \rho \tilde{\phi}_2^2) + \frac{1}{2}(\partial \tilde{y}^{m'})^2, \end{aligned} \quad (2.55)$$

where $\nabla_i = \partial_i - \omega_i$ is the conventional AdS_2 covariant derivative with ω_i defined in (2.53). We see that for this solution there are two bosonic modes of (non-constant) “mass” $\cosh \rho$ and three modes of “mass” $\sqrt{2} \cosh \rho$, with ρ being the classical solution $\rho(\sigma)$. The sum of the squared masses being

$$\sum m_b^2 = (2 + 3 \times 2) \cosh^2 \rho = 8 \cosh^2 \rho. \quad (2.56)$$

Using (2.51) and (2.53) in the definition of the projectors (2.16) we find

$$\not{e}_{\pm} = \cosh \rho \Gamma_{\pm}, \quad \Gamma_{\pm} = \Gamma_0 \pm \Gamma_1, \quad P_{\pm} = \frac{1}{2}(1 \pm \Gamma_{01}). \quad (2.57)$$

The GS fermionic Lagrangian (2.19) then reduces to

$$\mathcal{L}_{\text{GS}} = -i \cosh \rho (\Theta_+^1 \Gamma_- \partial_+ \Theta_+^1 + \Theta_-^3 \Gamma_+ \partial_- \Theta_-^3 + 4 \cosh \rho \Theta_+^1 \Gamma_{234} \Theta_-^3). \quad (2.58)$$

Note that the terms with the AdS_2 spin connection $\omega^{01} \Gamma_{01}$ vanish in (2.58), since $\Gamma_{\pm} \Gamma_{01} = \mp \Gamma_{\pm}$ is a symmetric matrix. For the same reason the overall $\cosh \rho$ factor can be removed by re-scaling Θ^I . Here $\Theta_+^1 = P_+ \Theta^1$ and $\Theta_-^3 = P_- \Theta^3$ are $8 + 8$ fermionic modes which represent 8 physical degrees of freedom of “mass” $\cosh \rho$. One can check that the UV divergences (proportional to sum of mass-squared terms) cancel between the bosonic modes (see (2.56)) and the fermionic modes in (2.58). We are effectively assuming that fluctuation operators are defined with respect to flat fiducial metric rather than the curved induced one so the conformal ghost contribution is trivial, cf. the discussion in [23].

In the pure-spinor action written for the present solution the fermions Θ_+^1 and Θ_-^3 completely decouple from the rest, and the Lagrangian (2.33), in which we replace the first line with its first-order counterpart (2.34), takes the following form

$$\begin{aligned} \mathcal{L}_{\text{PS}} = & i \cosh \rho (\Psi_-^1 \Gamma_+ \partial_- \Psi_-^1 + \Psi_+^3 \Gamma_- \partial_+ \Psi_+^3 - 4 \cosh \rho \Psi_-^1 \Gamma_{234} \Psi_+^3) \\ & - i \cosh \rho (\tilde{\Theta}_-^1 \Gamma_+ \partial_- \tilde{\Theta}_-^1 + \tilde{\Theta}_+^3 \Gamma_- \partial_+ \tilde{\Theta}_+^3) - 4i \nabla_- \Theta_+^1 \Gamma_{234} \nabla_+ \Theta_-^3. \end{aligned} \quad (2.59)$$

Here the first line is equivalent to the GS Lagrangian (2.58) and thus produces the same contribution to the partition function, while the fermions $\tilde{\Theta}_-^1$ and $\tilde{\Theta}_+^3$ in the second line are obviously massless.

To analyse the contribution of the remaining Θ_+^1 and Θ_-^3 fermions into the partition function, let us perform the transformation (2.32) with $\omega_{\pm} = -2 \sinh \rho \Gamma_{01}$ so that

$$e^{\frac{1}{4} \int d\xi^+ \omega_+} = e^{\frac{1}{4} \int d\xi^- \omega_-} = e^{-\frac{1}{2} \ln \cosh \rho \Gamma_{01}}.$$

Then the last term in (2.59) takes the form

$$\mathcal{L}' = -4i \nabla_- \Theta_+^1 \Gamma_{234} \nabla_+ \Theta_-^3 \rightarrow -4i \cosh \rho \partial_- \Theta_+^1 \Gamma_{234} \partial_+ \Theta_-^3. \quad (2.60)$$

The integration over Θ_+^1 and Θ_-^3 (done, e.g., by first changing the variables to $\partial_- \Theta_+^1 \rightarrow Y_1$, $\partial_+ \Theta_-^3 \rightarrow Y_3$ and assuming the local contribution of the integral over Y_1, Y_3 is canceled against a

¹²The bosonic as well as GS fermionic fluctuation Lagrangians can be found from the general folded string conformal gauge expressions in [24] by setting $\omega = 0$ therein.

factor in the path integral measure, cf. [29]) produces a massless determinant $(-\partial_+ \partial_-)^{16} = (\partial^2)^{16}$. Then the total fermionic contribution to the partition function in the PS model (2.59) is $[\det(\partial^2 - \cosh^2 \rho)]^8 (\det \partial^2)^{24}$, which matches the corresponding GS result (extra massless determinants are compensated by the ghosts as discussed above).¹³

Let us elaborate on the point that the treatment of the integral over Θ_+^1 and Θ_-^3 depends on assumptions about the definition of the corresponding path integral. *I.e.* the statement of equivalence between the GS and PS models assumes a particular prescription for the measure. Observing that in the present case of curved induced geometry $\nabla_+ \nabla_- - \nabla_- \nabla_+ = \cosh^2 \rho \Gamma_{01}$ and integrating by parts in the last term in (2.59) or in (2.60) we may rewrite it as

$$\mathcal{L}' = -4i \nabla_- \Theta_+^1 \Gamma_{234} \nabla_+ \Theta_-^3 = -4i \nabla_+ \Theta_+^1 \Gamma_{234} \nabla_- \Theta_-^3 + 4i \cosh^2 \rho \Theta_+^1 \Gamma_{234} \Theta_-^3. \quad (2.61)$$

Introducing the Lagrange multipliers Λ_+^1 and Λ_-^3 we may convert this into 1-st order form as

$$\begin{aligned} \mathcal{L}'' &= i \cosh \rho (-2\Lambda_+^1 \Gamma_- \nabla_+ \Theta_+^1 - 2\Lambda_-^3 \Gamma_+ \nabla_- \Theta_-^3 + 4 \cosh \rho \Lambda_+^1 \Gamma_{234} \Lambda_-^3 + 4 \cosh \rho \Theta_+^1 \Gamma_{234} \Theta_-^3) \\ &= i \cosh \rho (Y_+^1 \Gamma_- \partial_+ Y_+^1 + Y_-^3 \Gamma_+ \partial_- Y_-^3 + 4 \cosh \rho Y_+^1 \Gamma_{234} Y_-^3) \\ &\quad - i \cosh \rho (X_+^1 \Gamma_- \partial_+ X_+^1 + X_-^3 \Gamma_+ \partial_- X_-^3 - 4 \cosh \rho X_+^1 \Gamma_{234} X_-^3), \end{aligned} \quad (2.62)$$

where $X_\pm^{1,3} = \frac{1}{\sqrt{2}}(\Lambda_\pm^{1,3} + \Theta_\pm^{1,3})$ and $Y_\pm^{1,3} = \frac{1}{\sqrt{2}}(\Lambda_\pm^{1,3} - \Theta_\pm^{1,3})$. Since the last two lines here look like copies of the GS fermionic Lagrangian (2.58) one might naively conclude that (taking also into account the first line of (2.59)) the PS Lagrangian has three times more “massive” fermionic modes than the GS one. However, this conclusion is premature as it depends on the assumption that the introduction of the auxiliary fields in (2.62) did not produce additional determinant factors. This is not so in general as they enter into (2.62) with non-trivial background-dependent factors, so the final result depends on a proper definition of path integral measure.

2.4.2 Long spinning string in $AdS_3 \subset AdS_5$

Let us now consider the long infinite-spin string solution (2.52). The non-zero components of the pull-backs of the vielbeins to the worldsheet here are

$$\begin{aligned} e_\tau^0 &= \kappa \cosh \rho, & e_\tau^4 &= \kappa \sinh \rho, & e_\sigma^1 &= \kappa, \\ e_\tau^A \Gamma_A &= \kappa (\cosh \rho \Gamma_0 + \sinh \rho \Gamma_4), & e_\sigma^A \Gamma_A &= \kappa \Gamma_1, \end{aligned} \quad (2.63)$$

and the induced metric is flat $ds^2 = \kappa^2(-d\tau^2 + d\sigma^2)$. The Lagrangian for the (appropriately redefined) bosonic fluctuations around this solution ([24]) has the following form

$$\begin{aligned} \mathcal{L}_{\text{bose}} &= -\frac{1}{2}(\partial \tilde{t})^2 + \frac{1}{2}(\partial \tilde{\rho})^2 + \frac{1}{2}(\partial \tilde{\phi}_2)^2 + 4\kappa \tilde{t} \partial_\sigma \tilde{\phi}_2 - 4\kappa \tilde{\rho} \partial_\tau \tilde{\phi}_2 \\ &\quad + \frac{1}{2}((\partial \tilde{\theta})^2 + 2\kappa^2 \tilde{\theta}^2) + \frac{1}{2}((\partial \tilde{\phi}_1)^2 + 2\kappa^2 \tilde{\phi}_1^2) + \frac{1}{2}(\partial \tilde{y}^{m'})^2. \end{aligned} \quad (2.64)$$

It effectively describes two modes of mass $\sqrt{2}\kappa$, one mode of mass 2κ and seven massless modes. Their mass squared sum is $8\kappa^2$.

To write down the GS fermionic action let us note that the relevant components of the spin connection (2.3) are

$$\omega^{01} = -\sinh \rho dt, \quad \omega^{14} = \sin \theta \cosh \rho d\phi_2, \quad \omega^{24} = \cos \theta d\phi_2, \quad (2.65)$$

i.e., since $\theta = \frac{\pi}{2}$, we get $-\frac{1}{4}\omega_\tau^{AB}\Gamma_{AB} = \frac{\kappa}{2}(\sinh \rho \Gamma_0 + \cosh \rho \Gamma_4)\Gamma_1$. Thus one can perform the Lorentz rotation

$$\begin{aligned} \Theta &\rightarrow e^{\frac{\kappa}{2}\Gamma_{04}}\Theta, & e^{-\frac{\kappa}{2}\Gamma_{04}}(\sinh \rho \Gamma_0 + \cosh \rho \Gamma_4)e^{\frac{\kappa}{2}\Gamma_{04}} &\rightarrow \Gamma_4, \\ & & e^{-\frac{\kappa}{2}\Gamma_{04}}(\sinh \rho \Gamma_4 + \cosh \rho \Gamma_0)e^{\frac{\kappa}{2}\Gamma_{04}} &\rightarrow \Gamma_0, \end{aligned} \quad (2.66)$$

¹³The total result for the partition function of the infinite straight string surface should be trivial as discussed in [23].

under which the fermion covariant derivative becomes

$$\nabla_\tau = \partial_\tau + \frac{\kappa}{2}\Gamma_{41}, \quad \nabla_\sigma = \partial_\sigma + \frac{\kappa}{2}\Gamma_{04}, \quad \nabla = \eta^{ij}e_i^A\Gamma_A\nabla_j = \kappa(-\Gamma_0\partial_\tau + \Gamma_1\partial_\sigma) = \kappa\partial. \quad (2.67)$$

Using (2.66) and (2.67) the corresponding fermionic part of the GS Lagrangian (2.11) takes the following form

$$\mathcal{L}_{\text{GS}} = 2i(\Theta_+\not{\partial}\Theta_+ + \kappa\Theta_+\Gamma_{234}\sigma_1\Theta_+), \quad (2.68)$$

where $\Theta_+ = \frac{1}{2}(1 + \Gamma_{01}\sigma_3)\Theta$ are 16 component fermions which carry 8 physical degrees of freedom of mass κ . Their mass squared sum exactly cancels the mass squared sum of the bosonic modes (2.64).

After performing the Lorentz rotation (2.66) and (2.67), the quadratic fermionic part of the PS Lagrangian (2.24) takes the form¹⁴

$$\mathcal{L}_{\text{PS}} = -2\eta^{ij}\nabla_i\Theta\Gamma_{01234}\sigma_2\nabla_j\Theta - i\kappa^2\Theta\Gamma_{234}\sigma_1\Theta - i\kappa\Theta(1 + \Gamma_{01}\sigma_3)\not{\partial}\Theta. \quad (2.69)$$

Substituting the explicit expressions (2.67) for ∇_i and integrating by parts we get

$$\begin{aligned} \mathcal{L}_{\text{PS}} = & 2\Theta\Gamma_{01234}\sigma_2\partial_i\partial^i\Theta - i\kappa^2\Theta\Gamma_{234}\sigma_1(1 - \Gamma_{01}\sigma_3)\Theta \\ & - i\kappa\Theta(1 + \Gamma_{01}\sigma_3)\not{\partial}\Theta - 2\kappa\Theta\Gamma_{023}\sigma_2\partial_\tau\Theta - 2\kappa\Theta\Gamma_{123}\sigma_2\partial_\sigma\Theta. \end{aligned} \quad (2.70)$$

The direct computation of the determinant of the corresponding second-order kinetic operator D_{PS} gives

$$\det D_{\text{PS}} = [\det(\partial^2 - \kappa^2)]^8 (\partial^2)^{24}. \quad (2.71)$$

Thus the pure spinor fermion massive contribution is the same (8 fermionic modes of mass κ) as the GS one.

3 $AdS_2 \times S^2 \times T^6$ superstring

There are several $AdS_2 \times S^2 \times T^6$ backgrounds in type IIA and IIB string theories that are supported by RR fluxes (see [11] for a review and references). They preserve only 8 of 32 supersymmetries and are invariant under the superisometry group $PSU(1,1|2)$. The dynamics of a superstring whose motion is restricted to the four-dimensional subspace $AdS_2 \times S^2$ can be described by two different $\frac{PSU(1,1|2)}{SO(1,1) \times U(1)}$ supercoset sigma-models with 8 fermionic fields. The first one is of the Green-Schwarz type [34], having similar structure to the $AdS_5 \times S^5$ GS superstring [3], and the second one is an $N = (2, 2)$ worldsheet superconformal $\frac{PSU(1,1|2)}{SO(1,1) \times U(1)}$ sigma-model [5] which is the $AdS_2 \times S^2$ counterpart of the supercoset sector of the $AdS_5 \times S^5$ pure-spinor model.

When extended to the whole $AdS_2 \times S^2 \times T^6$ space-time, the GS superstring sigma-model gets enlarged with 6 bosons and 24 fermions which couple in a non-trivial way to the $4d \frac{PSU(1,1|2)}{SO(1,1) \times U(1)}$ supercoset sector [11]. On the other hand, the $\frac{PSU(1,1|2)}{SO(1,1) \times U(1)}$ sigma-model of [5] gets enlarged with an $N = (2, 2)$ worldsheet superconformal Ramond-Neveu-Schwarz T^6 -sector which couples to the supercoset sector only indirectly via the (super)Virasoro constraints. The latter model is called hybrid [35, 5] since it is constructed using both the target-space spinors as in the GS formulation and the worldsheet spinors as in the RNS formulation.

The purely bosonic sectors of the GS superstring and the hybrid model for $AdS_2 \times S^2 \times T^6$ are the same and in the conformal gauge have the form

$$\mathcal{L}_{\text{bose}} = \frac{1}{2}\eta^{ij}(g_{ij}^{AdS_2} + g_{ij}^{S^2} + g_{ij}^{T^6}) = \frac{1}{2}\eta^{ij}G_{ij}(X), \quad (3.1)$$

where $G_{ij}(X)$ ($X^M = (x^m, x^{\hat{m}}, y^{m'})$) is the induced worldsheet metric, $g_{ij}^{AdS_2} = g_{mn}^{AdS_2}(x)\partial_i x^m \partial_j x^n$, $g_{ij}^{S^2} = g_{\hat{m}\hat{n}}^{S^2}(\hat{x})\partial_i x^{\hat{m}} \partial_j x^{\hat{n}}$ and $g_{ij}^{T^6} = g_{a'b'}^{T^6}\partial_i y^{a'} \partial_j y^{b'}$.

¹⁴Because of the simplicity of the form of the PS Lagrangian in this background, here we do not need to split Θ into Θ_\pm^1 and Θ_\pm^3 as in (2.25).

The Virasoro constraints and the bosonic equations of motion are as in (2.4) and (2.6) in which e_j^A are now the worldsheet pullbacks of the $AdS_2 \times S^2 \times T^6$ vielbeins $d\xi^i \partial_i X^M e_M^A(X)$ ($A = a, \hat{a}, a'$).

The fermionic sectors of the two models are significantly different and to find the direct relation between them, which may involve a non-trivial (non-linear and non-local) change of variables is an open problem. In what follows we shall study the relation between the two formulations of the $AdS_2 \times S^2 \times T^6$ superstring by comparing their quadratic fermionic actions (and thus one-loop semiclassical partition functions), similarly to what was done in the previous section for the GS and PS formulations of the $AdS_5 \times S^5$ superstring.

3.1 Fermionic part of the GS action

The quadratic fermionic part of the GS Lagrangian on $AdS_2 \times S^2 \times T^6$ in conformal gauge can be written as¹⁵

$$\mathcal{L}_{\text{GS}} = i\Theta (\eta^{ij} - \varepsilon^{ij}\Gamma_{11}) e_i^A \Gamma_A \left(\nabla_j + \frac{1}{2} \mathcal{P}_8 \gamma \Gamma_{11} \Gamma_B e_j^B \right) \Theta, \quad (3.2)$$

where Θ is a 32 component Majorana spinor, $\gamma = \Gamma^{01}$ and \mathcal{P}_8 is a projector of rank 8, whose presence implies that in $AdS_2 \times S^2 \times T^6$ the 32-component supersymmetry of the $D = 10$ type IIA vacuum is broken down to 8 (see [11] for more details). Splitting Θ as¹⁶ $\Theta^1 = \frac{1}{2}(1 + \Gamma_{11})\Theta$ and $\Theta^3 = \frac{1}{2}(1 - \Gamma_{11})\Theta$ we get

$$\mathcal{L}_{\text{GS}} = -i\Theta^1 \not\! \partial_+ \Theta^1 - i\Theta^3 \not\! \partial_- \Theta^3 - i\Theta^1 \not\! \partial_+ \mathcal{P}_8 \gamma \not\! \partial_- \Theta^3. \quad (3.3)$$

Since the bosonic vielbeins satisfy the on-shell equations (2.6), (2.4), (2.7), the matrices $\not\! \partial_\pm = e_\pm^A \Gamma_A$ are covariantly (anti)holomorphic and square to zero like in (2.14). As in the $AdS_5 \times S^5$ case, the kappa-symmetry of the GS formulation manifests itself in the fact that only half of the 32 fermions $\Theta^{1,3}$, projected respectively with $\not\! \partial_+$ and $\not\! \partial_-$, appear in the action (3.3).

It should be noted that the components of Θ projected with \mathcal{P}_8 and $\mathcal{P}_{24} = 1 - \mathcal{P}_8$ have different geometrical meaning. Eight fermions $\vartheta = \mathcal{P}_8 \Theta$ parametrize the Grassmann-odd directions of the supercoset $\frac{PSU(1,1|2)}{SO(1,1) \times U(1)}$, while twenty four fermions $v = (1 - \mathcal{P}_8)\Theta = \mathcal{P}_{24}\Theta$ correspond to the 24 broken target-space supersymmetries. In the Lagrangian ϑ and v couple to each other through the bosonic modes in T^6 as can be seen by rewriting (3.2) in the more explicit form (see [11] for details)

$$\begin{aligned} \mathcal{L}_{\text{GS}} = & i\vartheta (\eta^{ij} - \varepsilon^{ij}\Gamma_{11}) e_i^{\underline{a}} \Gamma_{\underline{a}} \mathcal{D}_j \vartheta \\ & + i\vartheta (\eta^{ij} - \varepsilon^{ij}\Gamma_{11}) \Gamma_{a'} \nabla_j v \partial_i y^{a'} + iv (\eta^{ij} - \varepsilon^{ij}\Gamma_{11}) \Gamma_{a'} \nabla_j \vartheta \partial_i y^{a'} \\ & + \frac{i}{2} \vartheta (\eta^{ij} - \varepsilon^{ij}\Gamma_{11}) e_i^{\underline{a}} \Gamma_{\underline{a}} \mathcal{P}_8 \gamma \Gamma_{11} \Gamma_{a'} v \partial_j y^{a'} \\ & + iv (\eta^{ij} - \varepsilon^{ij}\Gamma_{11}) (e_i^{\underline{a}} \Gamma_{\underline{a}} + e_i^{a'} \Gamma_{a'}) \nabla_j v \\ & + \frac{i}{2} v (\eta^{ij} - \varepsilon^{ij}\Gamma_{11}) \Gamma_{a'} \mathcal{P}_8 \gamma \Gamma_{11} \Gamma_{b'} v \partial_i y^{a'} \partial_j y^{b'}, \end{aligned} \quad (3.4)$$

where

$$\nabla_j = \partial_j - \frac{1}{4} \Gamma_{\underline{ab}} \omega_j^{\underline{ab}}(x), \quad \mathcal{D}_j = \nabla_j + \frac{1}{2} \mathcal{P}_8 \gamma \Gamma_{11} \Gamma_{\underline{b}} e_j^{\underline{b}} \quad (3.5)$$

and $\omega_i^{\underline{ab}}(x)$ and $e_i^{\underline{a}}(x)$ ($\underline{a}, \underline{b} = 0, 1, 2, 3$) are the worldsheet pull-backs of the spin connection and the local frame in $AdS_2 \times S^2$. Note that the $AdS_2 \times S^2$ and T^6 sectors are coupled via the couplings between ϑ and v .

¹⁵In what follows we consider the case of the type IIA $AdS_2 \times S^2 \times T^6$ supergravity background discussed in Section 3.1 of [11].

¹⁶As in the $AdS_5 \times S^5$ case, the labels 1, 3 refer to the \mathbb{Z}_4 -grading of the underlying supercoset $\frac{PSU(1,1|2)}{SO(1,1) \times SO(2)}$.

3.2 Fermionic part of the hybrid model action

The hybrid model in $AdS_2 \times S^2 \times T^6$ consists of a supercoset sigma-model on $\frac{PSU(1,1|2)}{SO(1,1) \times U(1)}$ (similar to that in the pure-spinor formulation of the $AdS_5 \times S^5$ superstring) and an RNS-like string model on T^6 [5]. In contrast to the GS superstring, in the hybrid model action the $\frac{PSU(1,1|2)}{SO(1,1) \times U(1)}$ sector containing $AdS_2 \times S^2$ is completely decoupled from the T^6 sector.

In the conventions of [11], the $AdS_2 \times S^2$ supercoset part of the hybrid model restricted to the second order in the eight coset fermions $\vartheta = \mathcal{P}_8 \Theta$ has the following form

$$\begin{aligned} \mathcal{L}_H = & i\vartheta (\eta^{ij} - \varepsilon^{ij} \Gamma_{11}) e_i^{\underline{a}} \Gamma_{\underline{a}} \mathcal{D}_j \vartheta \\ & + 2i\eta^{ij} \mathcal{D}_i \vartheta \gamma \Gamma_{11} \mathcal{D}_j \vartheta, \end{aligned} \quad (3.6)$$

where there is no need to include the projector \mathcal{P}_8 in \mathcal{D}_j in (3.5) as it commutes with $\Gamma_{\underline{a}}$.

In the T^6 sector the hybrid model contains six RNS-like 2-component worldsheet spinors $\Psi^{a'}$

$$L_\Psi = \frac{i}{2} \Psi^{a'} \gamma^i \partial_i \Psi^{a'}, \quad (3.7)$$

where γ^i is a 2-dimensional gamma-matrix. The model also includes a ghost sector which consists of a chiral and an anti-chiral boson. The whole construction possesses $N = (2, 2)$ worldsheet superconformal symmetry for which the total central charge is zero (see [5] for more details). As we have already mentioned, in this model the $AdS_2 \times S^2$ supercoset sector is decoupled from the T^6 one (apart from indirect relation via the $N = (2, 2)$ super-Virasoro constraints).

The first line in (3.6) is the same as the supercoset GS fermion term in the first line of eq.(3.4) and the second line is similar to the second-derivative fermion term in the PS action. Adding a total derivative term, the Lagrangian (3.6) can be rewritten in the following form

$$\mathcal{L}_H = 2i \nabla_i \vartheta \gamma \Gamma_{11} (\eta^{ij} - \varepsilon^{ij} \Gamma_{11}) (\nabla_j \vartheta + \frac{1}{2} \gamma \Gamma_{11} \Gamma_{\underline{b}} e_j^{\underline{b}} \vartheta). \quad (3.8)$$

Introducing $\vartheta^1 = \frac{1}{2}(1 + \Gamma_{11})\vartheta$ and $\vartheta^3 = \frac{1}{2}(1 - \Gamma_{11})\vartheta$ we get

$$\mathcal{L}_H = -4i \nabla_+ \vartheta^1 \Gamma_{01} \nabla_- \vartheta^3 + i\vartheta^1 e_-^{\underline{a}} \Gamma_{\underline{a}} \nabla_+ \vartheta^1 + i\vartheta^3 e_+^{\underline{a}} \Gamma_{\underline{a}} \nabla_- \vartheta^3, \quad (3.9)$$

where due to the bosonic equations of motion (2.6) and the Virasoro constraints (2.4) we have

$$\nabla_+ (e_-^{\underline{a}} \Gamma_{\underline{a}}) = 0 = \nabla_- (e_+^{\underline{a}} \Gamma_{\underline{a}}), \quad (e_\pm^{\underline{a}} \Gamma_{\underline{a}})^2 = -(e_\pm^{a'} \Gamma_{a'})^2. \quad (3.10)$$

Here $e_\pm^{a'}(y) = \partial_\pm y^{a'}$ are pullbacks of the T^6 vielbeins.

Comparing eq. (3.9) with (the supercoset part of) the GS action (3.3) we see (as in the pure-spinor formulation of Section 2) that their first-derivative terms are similar up to the interchange of ϑ^1 and ϑ^3 , while the mass-like term of the GS action in the hybrid model gets replaced by the second-derivative term.

The difference of the hybrid model from the PS superstring is that in the hybrid model in addition to the supercoset sector we have the T^6 sector which includes RNS-like fermions. So we should distinguish two cases: (i) When the classical string moves entirely in $AdS_2 \times S^2$ (i.e. $e_\pm^{a'}(y) = \partial_\pm y^{a'} = 0$) and (ii) when the string also moves in T^6 .

In the first case $(e_\pm^{a'} \Gamma_{a'})^2$ vanishes and the problem reduces to the analysis of the supercoset part of the hybrid model. This can be done in the same way as for the PS superstring in Section 2.2. One then concludes that around a generic classical string solution in $AdS_2 \times S^2$ the hybrid model has 2 pairs of supercoset fermionic modes that produce the same functional determinant as the GS supercoset fermions.

However, the matching of the contributions of the other fermionic modes and the ghosts in the two models turns out to be less straightforward than in the GS vs pure-spinor case. Already in flat space, to relate the hybrid model to the GS model in the light-cone gauge it is necessary to bosonise 2 of the 6 extra pairs of supercoset hybrid-model fermions into an additional pair of chiral

and anti-chiral scalars [35]. A similar procedure is expected to apply in a curved background as well.¹⁷

Assuming this is the case, the comparison of the contributions into the 1-loop partition functions in the two formulations goes as follows. The bosonic sectors of the two models coincide. In the GS formulation 2 pairs of supercoset fermionic modes ϑ contribute to the partition function with $\det \nabla_{\text{GS}}$. The 6 pairs of non-coset fermions v give $(\det \partial^2)^6$ and the conformal ghosts give $(\det \partial^2)^2$, in total $(\det \partial^2)^6 (\det \partial^2)^2 = (\det \partial^2)^8$. In the hybrid model the 2 pairs of the supercoset fermionic modes produce the same determinant as in the GS case, i.e. $\det \nabla_{\text{GS}}$. The one chiral and one anti-chiral scalar ghost [5] give $(\det \partial^2)^{-1}$. Two of the extra 6 coset fermions should be bosonised into an additional pair of chiral and anti-chiral scalars which give an extra $(\det \partial^2)^{-1}$, while the remaining 4 pairs of coset fermions and the RNS-like fermions $\Psi^{a'}$ give $(\det \partial^2)^{10}$. Putting all the contributions together we find that the resulting hybrid model partition function coincides with the one of the GS model.

The case of generic motion of the string in $AdS_2 \times S^2 \times T^6$ is more complicated (since, in particular, the Virasoro constraints (3.10) relate the $AdS_2 \times S^2$ and T^6 sectors) and it should be treated separately. In what follows we shall demonstrate that the matching of the “massive” fermionic modes take place at least in two simple examples – for a BMN-type geodesic running along S^2 and T^6 and for a classical string wrapping a circle in T^6 . We will also compare the GS and hybrid model fermionic actions in the background of world sheet instantons.

3.3 Examples of equivalence

3.3.1 Expansion near BMN geodesic

BMN limits for $AdS_p \times S^q$ backgrounds and their supersymmetries were considered, e.g., in [36]. In $AdS_2 \times S^2 \times T^6$ there are different ways to take the BMN limit. One way is to consider a geodesic running along S^2 but a more general possibility is to take a geodesic along both S^2 and T^6 . These cases were considered in detail in [11]. Let us choose the metric of $AdS_2 \times S^2 \times T^6$ in the form (we set the radii to 1)

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + d\theta^2 + \sin^2 \theta d\varphi^2 + dy_{a'} dy^{a'}. \quad (3.11)$$

If the geodesic representing the center of mass of the string runs along a “diagonal” direction in the $S^1 \times S^1$ torus formed by the equator $S^1 \subset S^2$ (with coordinate φ) and one of the $S^1 \subset T^6$ directions, e.g., y^4 , it can be parameterized by the “rotated” coordinate φ' as

$$t = m\tau, \quad \varphi' = m\tau, \quad \varphi' = \cos \alpha \varphi + \sin \alpha y^4, \quad y^{4'} = -\sin \alpha \varphi + \cos \alpha y^4, \quad (3.12)$$

where α is related to the ratio of string angular momenta along the two circles. $\alpha = 0$ is the case when the geodesic runs along φ and of S^2 and $\alpha = \frac{\pi}{2}$ corresponds to a geodesic running along $S^1 \subset T^6$. The resulting quadratic Lagrangian of the bosonic fluctuations is

$$\mathcal{L}_{\text{bose}} = \frac{1}{2} \partial_i \tilde{\rho} \partial^i \tilde{\rho} + \frac{1}{2} \partial_i \tilde{\theta} \partial^i \tilde{\theta} + \frac{m^2}{2} \tilde{\rho}^2 + \frac{1}{2} m^2 \cos^2 \alpha \tilde{\theta}^2 + \frac{1}{2} \partial_i y^{a'} \partial^i y_{a'}, \quad (3.13)$$

i.e. $\tilde{\rho}$ has mass m and $\tilde{\theta}$ has mass $m \cos \alpha$.

The GS Lagrangian for the fermionic fluctuations around the generic BMN solution has the following form in the light-cone kappa-symmetry gauge $\Gamma^+ \Theta = 0$

$$\begin{aligned} \mathcal{L}_{\text{GS}} = & -i\Theta^1 \Gamma^- \partial_- \Theta^1 - i\Theta^3 \Gamma^- \partial_+ \Theta^3 + im(1 + \cos \alpha) \tilde{\vartheta}^1 \Gamma^- \Gamma^r \tilde{\vartheta}^3 \\ & + im(1 - \cos \alpha) \tilde{v}^1 \Gamma^- \Gamma^r \tilde{v}^3, \end{aligned} \quad (3.14)$$

where the labels 1 and 3 refer again to the \mathbb{Z}_4 -grading (i.e. the chirality) and Γ^r corresponds to the radial direction of AdS_2 . The spinors $\tilde{\vartheta}$ and \tilde{v} are four-component fermions among $\Theta^{1,3}$

¹⁷We thank Nathan Berkovits for comments on this issue.

which carry two physical degrees of freedom each, having, respectively, mass $\frac{m}{2}(1 + \cos \alpha)$ and $\frac{m}{2}(1 - \cos \alpha)$ (see [11] for more details). The sums of the mass squared terms for the bosonic and fermionic modes match in agreement with UV finiteness of the model.

Let us now turn to the hybrid model and start with the $\alpha = 0$ case (i.e. when the geodesic runs inside S^2). The corresponding BMN limit of the hybrid model was considered in [26]. Here there is no kappa-symmetry, so we cannot impose $\Gamma^+ \vartheta = 0$. Let us denote half of ϑ which satisfies $\Gamma^+ \vartheta = 0$ by $X^1 = \frac{1}{2}(1 + \Gamma_{11})X^1$, $X^3 = \frac{1}{2}(1 - \Gamma_{11})X^3$ and the other half by Y^1, Y^3 , i.e.

$$(X^1, X^3) = \Gamma^+ \Gamma^- \vartheta, \quad (Y^1, Y^3) = \Gamma^- \Gamma^+ \vartheta. \quad (3.15)$$

To the leading order we consider we may replace $e_i \underline{a} \Gamma_{\underline{a}} \rightarrow -2m \delta_i^0 \Gamma^-$. Then from (3.9) we find

$$\begin{aligned} \mathcal{L}_H = & -2im X^1 \Gamma^- \partial_+ X^1 - 2im X^3 \Gamma^- \partial_- X^3 + 4i \partial_+ X^1 \Gamma^- \Gamma^r \partial_- X^3 \\ & + 4i \partial_+ Y^1 \Gamma^+ \Gamma^r \partial_- Y^3, \end{aligned} \quad (3.16)$$

where Γ^r corresponds again to the spatial direction of AdS_2 . Written in first order form for the X -fermions (3.16) reads

$$\begin{aligned} \mathcal{L}_H = & -i\psi^1 \Gamma^- \Gamma^r \psi^3 + 2i\psi^1 \Gamma^- \Gamma^r \partial_- X^3 + 2i \partial_+ X^1 \Gamma^- \Gamma^r \psi^3 \\ & - 2im X^1 \Gamma^- \partial_+ X^1 - 2im X^3 \Gamma^- \partial_- X^3 + 4i \partial_+ Y^1 \Gamma^+ \Gamma^r \partial_- Y^3. \end{aligned} \quad (3.17)$$

The redefinitions

$$X^1 = \frac{1}{2\sqrt{m}}(\tilde{X}^1 - \Gamma^r \tilde{\vartheta}^3), \quad X^3 = \frac{1}{2\sqrt{m}}(\tilde{X}^3 + \Gamma^r \tilde{\vartheta}^1), \quad \psi^1 = \sqrt{m} \tilde{\vartheta}^1, \quad \psi^3 = \sqrt{m} \tilde{\vartheta}^3 \quad (3.18)$$

lead to

$$\begin{aligned} \mathcal{L}_H = & \frac{i}{2} \tilde{\vartheta}^1 \Gamma^- \partial_- \tilde{\vartheta}^1 + \frac{i}{2} \tilde{\vartheta}^3 \Gamma^- \partial_+ \tilde{\vartheta}^3 - im \tilde{\vartheta}^1 \Gamma^- \Gamma^r \tilde{\vartheta}^3 - \frac{i}{2} \tilde{X}^1 \Gamma^- \partial_+ \tilde{X}^1 - \frac{i}{2} \tilde{X}^3 \Gamma^- \partial_- \tilde{X}^3 \\ & + 4i \partial_+ Y^1 \Gamma^+ \Gamma^r \partial_- Y^3. \end{aligned} \quad (3.19)$$

We read off that $\tilde{\vartheta}^{1,3}$ are two physical fermionic modes of mass m . Since $\Gamma^+ \tilde{\vartheta} = 0$ these match exactly the massive modes of the GS string in (3.14).

In the case of more general geodesic along S^2 and T^6 the action (3.8) reduces to

$$\begin{aligned} \mathcal{L}_H = & -i(1 + \cos \alpha)m X^1 \Gamma^- \partial_+ X^1 - i(1 + \cos \alpha)m X^3 \Gamma^- \partial_- X^3 \\ & + 4i \partial_+ X^1 \Gamma^- \Gamma^r \partial_- X^3 - i(1 - \cos \alpha)m Y^1 \Gamma^+ \partial_+ Y^1 \\ & - i(1 - \cos \alpha)m Y^3 \Gamma^+ \partial_- Y^3 + 4i \partial_+ Y^1 \Gamma^+ \Gamma^r \partial_- Y^3, \end{aligned} \quad (3.20)$$

where $\Gamma^\pm = \frac{1}{2}(\Gamma^0 \pm \Gamma^3)$, with the index 3 denoting the φ direction of S^2 . The analysis of the spectrum follows the same lines as in eqs. (3.17)–(3.19). Passing to the first-order form we get

$$\begin{aligned} \mathcal{L}_H = & \frac{i}{2} \left[\tilde{\vartheta}^1 \Gamma^- \partial_- \tilde{\vartheta}^1 + \tilde{\vartheta}^3 \Gamma^- \partial_+ \tilde{\vartheta}^3 - i(1 + \cos \alpha)m \tilde{\vartheta}^1 \Gamma^- \Gamma^r \tilde{\vartheta}^3 \right. \\ & + \tilde{v}^1 \Gamma^+ \partial_- \tilde{v}^1 + \tilde{v}^3 \Gamma^+ \partial_+ \tilde{v}^3 - i(1 - \cos \alpha)m \tilde{v}^1 \Gamma^+ \Gamma^r \tilde{v}^3 \\ & \left. - \tilde{X}^1 \Gamma^- \partial_+ \tilde{X}^1 - \tilde{X}^3 \Gamma^- \partial_- \tilde{X}^3 - \tilde{Y}^1 \Gamma^+ \partial_+ \tilde{Y}^1 - \tilde{Y}^3 \Gamma^+ \partial_- \tilde{Y}^3 \right]. \end{aligned} \quad (3.21)$$

Thus two pairs of fermions $\tilde{\vartheta}^{1,3}$ have the mass $\frac{m}{2}(1 + \cos \alpha)$ and the two fermions $\tilde{v}^{1,3}$ have the mass $\frac{m}{2}(1 - \cos \alpha)$. These masses agree with those of the corresponding BMN limit of the GS superstring in (3.14), although all the GS fermions satisfy the condition $\Gamma^+ \Theta = 0$, while in the hybrid case ϑ and v are projected with Γ^\pm , respectively.

As in the pure-spinor model case, the number of massless fermions of the hybrid model do not appear to match that of the GS model, but as was already discussed in Section 3.2, the contribution of extra massless fermions of the hybrid model, upon the bosonization of two of them, should be compensated by the chiral boson ghosts so that at the end there is a match with the GS formulation.

3.3.2 String wrapping an $S^1 \subset T^6$

Next, let us now consider a simple solution in which the string wraps an S^1 circle of T^6 (we shall label it by index 9)

$$t = \tau, \quad y^9 = \sigma, \quad \theta = \frac{\pi}{2}, \quad \rho = \varphi = y^{4,5,6,7,8} = 0. \quad (3.22)$$

The corresponding induced worldsheet metric is flat. The Lagrangian of bosonic fluctuations is

$$\mathcal{L}_{\text{bose}} = -\frac{1}{2}(\partial\tilde{t})^2 + \frac{1}{2}[(\partial\tilde{\rho})^2 + \tilde{\rho}^2] + \frac{1}{2}(\partial\tilde{\theta})^2 + \frac{1}{2}(\partial\tilde{\varphi})^2 + \frac{1}{2}(\partial\tilde{y}^{a'})^2. \quad (3.23)$$

The quadratic fermionic part of the GS string Lagrangian (3.2) here simplifies to

$$\begin{aligned} \mathcal{L}_{\text{GS}} &= i\Theta(1 - \Gamma_{09}\Gamma_{11})\not{\partial}\Theta - \frac{i}{2}\Theta(1 - \Gamma_{09}\Gamma_{11})\mathcal{P}_8\Gamma_{01}\Gamma_{11}\Theta \\ &\quad - \frac{i}{2}\Theta(1 - \Gamma_{09}\Gamma_{11})\Gamma_9\mathcal{P}_8\Gamma_{01}\Gamma_{11}\Gamma_9\Theta, \end{aligned} \quad (3.24)$$

where $\not{\partial} = -\Gamma_0\partial_\tau + \Gamma_9\partial_\sigma$. Introducing $\Theta_\pm = \frac{1}{2}(1 \pm \Gamma_{09}\Gamma_{11})\Theta$ we get

$$\begin{aligned} \mathcal{L}_{\text{GS}} &= 2i\Theta_-\not{\partial}\Theta_- - 2i\Theta_-\mathcal{P}_8\Gamma_{01}\Gamma_{11}\Theta_- - i\Theta_-\mathcal{P}_8\Gamma_{01}\Gamma_{11}\Theta_+ + i\Theta_-\mathcal{P}_8\Gamma_{01}\Gamma_{11}\Theta_+ \\ &= 2i\Theta_-\Gamma_0(-\partial_\tau + \Gamma_{11}\partial_\sigma)\Theta_- - 2i\Theta_-\mathcal{P}_8\Gamma_{01}\Gamma_{11}\Theta_-. \end{aligned} \quad (3.25)$$

The fermions Θ_+ drop out of the Lagrangian which is again a manifestation of the κ -symmetry. We are thus left with 16 fermions Θ_- . In (3.25) their mass term contains the projector \mathcal{P}_8 ; hence, only $\mathcal{P}_8\Theta_-$ fermionic modes are massive. To compute the number of their components we should find the rank of the projector $\mathcal{P}_8(1 - \Gamma)$ where $\Gamma = \Gamma_{09}\Gamma_{11}$. Since $\mathcal{P}_8\Gamma_9\mathcal{P}_8 = 0$, we can rewrite the projector as follows

$$\mathcal{P}_8(1 - \Gamma) = \mathcal{P}_8 - \mathcal{P}_8\Gamma(\mathcal{P}_{24} + \mathcal{P}_8) = \mathcal{P}_8 - \mathcal{P}_8\Gamma\mathcal{P}_{24}, \quad (3.26)$$

where \mathcal{P}_{24} is the projector complementary to \mathcal{P}_8 . The projector complementary to (3.26) has the form

$$1 - \mathcal{P}_8(1 - \Gamma) = \mathcal{P}_{24} + \mathcal{P}_8\Gamma\mathcal{P}_{24} = (1 + \mathcal{P}_8\Gamma)\mathcal{P}_{24}. \quad (3.27)$$

The matrix $1 + \mathcal{P}_8\Gamma$ is invertible, its inverse being $1 - \mathcal{P}_8\Gamma$. Hence, the projector (3.27) has rank 24 and the projector (3.26) has rank 8. We thus conclude that $\mathcal{P}_8\Theta_-$ is an 8-component spinor which carries 4 independent physical fermionic modes.

We should now split the kinetic term of eq. (3.25) into two parts, one for the massive eight-component fermions $\tilde{\vartheta} = \mathcal{P}_8\Theta_-$ and another one for the massless 8-component fermions $\psi = (1 + \mathcal{P}_8\Gamma)\mathcal{P}_{24}\Theta_-$ which complement $\tilde{\vartheta}$ to the 16-component spinor $\frac{1}{2}(1 - \Gamma)\Theta$. To this end we insert into the kinetic term the unit matrix $1 = \mathcal{P}_8(1 - \Gamma) + (1 + \mathcal{P}_8\Gamma)\mathcal{P}_{24}$

$$\begin{aligned} 2i\Theta_-\Gamma_0(-\partial_\tau + \Gamma_{11}\partial_\sigma)\Theta_- &= 2i\Theta_-(\mathcal{P}_8(1 - \Gamma) + (1 + \mathcal{P}_8\Gamma)\mathcal{P}_{24})\Gamma_0(-\partial_\tau + \Gamma_{11}\partial_\sigma)\Theta_- \\ &= 4i\tilde{\vartheta}\Gamma_0(-\partial_\tau + \Gamma_{11}\partial_\sigma)\tilde{\vartheta} + 2i\psi\Gamma_0(-\partial_\tau + \Gamma_{11}\partial_\sigma)\psi. \end{aligned} \quad (3.28)$$

We end up with the Lagrangian

$$\mathcal{L}_{\text{GS}} = 4i\tilde{\vartheta}\Gamma_0(-\partial_\tau + \Gamma_{11}\partial_\sigma)\tilde{\vartheta} + 2i\tilde{\vartheta}\Gamma_{01}\Gamma_{11}\tilde{\vartheta} + 2i\psi\Gamma_0(-\partial_\tau + \Gamma_{11}\partial_\sigma)\psi. \quad (3.29)$$

Thus $\tilde{\vartheta}$ describes 4 physical fermionic modes of mass $\frac{1}{2}$, so that the boson-fermion mass-squared sum rule is again satisfied.

The fermionic part of the hybrid model Lagrangian (3.8) takes the form

$$\mathcal{L}_{\text{H}} = i\vartheta\Gamma_0(\partial_\tau + \Gamma_{11}\partial_\sigma)\vartheta - 2i\partial_i\vartheta\Gamma_{01}\Gamma_{11}\partial^i\vartheta. \quad (3.30)$$

In terms of $\vartheta_{\pm} = \frac{1}{2}(1 \pm \Gamma_{01}\Gamma_{11})\vartheta$ we get

$$\begin{aligned}\mathcal{L}_H &= -i\vartheta_+\not{\partial}\vartheta_+ - 2i\partial_i\vartheta_+\partial^i\vartheta_+ - i\vartheta_-\not{\partial}\vartheta_- + 2i\partial_i\vartheta_-\partial^i\vartheta_- \\ &= -i\vartheta_+\not{\partial}\vartheta_+ + 2i\partial_i\vartheta_+\partial^i\vartheta_+ - i\vartheta_-\not{\partial}\vartheta_- - 2i\partial_i\vartheta_-\partial^i\vartheta_-, \end{aligned}\quad (3.31)$$

where now $\not{\partial} = -\Gamma_0(\partial_\tau + \Gamma_{11}\partial_\sigma)$. We can now pass to the first order form by introducing the Lagrange multipliers ψ_{\pm}

$$\begin{aligned}\mathcal{L}_H &= -i\vartheta_+\not{\partial}\vartheta_+ + 2i\psi_+\not{\partial}\vartheta_+ - \frac{i}{2}\psi_+\psi_+ - i\vartheta_-\not{\partial}\vartheta_- - 2i\psi_-\not{\partial}\vartheta_- + \frac{i}{2}\psi_-\psi_- \\ &= i\tilde{\vartheta}_+\not{\partial}\tilde{\vartheta}_+ - i\tilde{\psi}_+\not{\partial}\tilde{\psi}_+ - \frac{i}{2}\tilde{\vartheta}_+\tilde{\vartheta}_+ + i\tilde{\vartheta}_-\not{\partial}\tilde{\vartheta}_- - i\tilde{\psi}_-\not{\partial}\tilde{\psi}_- + \frac{i}{2}\tilde{\vartheta}_-\tilde{\vartheta}_-, \end{aligned}\quad (3.32)$$

where $\tilde{\vartheta}_{\pm} = \psi_{\pm}$ and $\tilde{\psi}_{\pm} = \vartheta_{\pm} \mp \psi_{\pm}$. We thus get 4 fermionic modes $\tilde{\vartheta}_{\pm}$ of mass $\frac{1}{2}$ and 4 massless modes $\tilde{\psi}_{\pm}$ plus 6 massless fermions $\Psi^{a'}$ (3.7). The spectrum of massive fermions is again the same as for the GS superstring.

3.4 Expansion near S^2 and T^2 worldsheet instantons

Let us now discuss a less trivial example of a classical solution – worldsheet instantons which exist in the Wick-rotated superstring theory in this background, as well as in $AdS_4 \times CP^3$ [37]¹⁸. There are two types of instantons. One is when the string wraps S^2 . This solution is a Wick rotated counterpart of the AdS_2 -filling string solution considered in Section 2.4.1. For a (single) worldsheet instanton wrapping S^2 the geometry is that of the two-sphere, while the string coordinates along $AdS_2 \times T^6$ are constant. For the description of this solution [38] it is convenient to introduce complex coordinates both in the worldsheet and in the target space. Written in terms of a complex coordinate ζ on S^2

$$\zeta = \tan \frac{\theta}{2} e^{i\varphi}, \quad ds^2 = d\theta^2 + \sin^2 \theta d\varphi^2, \quad (3.33)$$

the conformal gauge action on S^2 reads

$$S_E = \int d^2z \frac{|\partial\zeta|^2 + |\bar{\partial}\zeta|^2}{(1 + |\zeta|^2)^2}. \quad (3.34)$$

This action has a local minimum if $\bar{\partial}\zeta = 0$ or $\partial\zeta = 0$, *i.e.* the embedding is given by a holomorphic function $\zeta = \zeta(z)$ for the instanton or by an anti-holomorphic function $\zeta = \zeta(\bar{z})$ for the anti-instanton.

Another type of worldsheet instanton is described by a Euclidean string worldsheet wrapping a T^2 in T^6 . In this case the worldsheet coordinates can be directly identified with those of the T^2 torus, $y^i = \xi^i$, $\xi = (\xi^1, \xi^2)$.

3.4.1 Green-Schwarz action

Let us consider the quadratic fermionic part of the GS action expanded near these instanton solutions.

S^2 instanton

In the S^2 instanton background the Wick rotated fermionic part of the GS action (3.4) reduces to

$$\begin{aligned}\mathcal{L}_{GS} &= i\vartheta(\sqrt{-h}h^{ij} + i\varepsilon^{ij}\Gamma_{11})e_i^{\hat{a}}\Gamma_{\hat{a}}(\nabla_j + \frac{1}{2}\mathcal{P}_8\gamma\Gamma_{11}\Gamma_{\hat{a}}e_j^{\hat{a}})\vartheta \\ &\quad + iv(\sqrt{-h}h^{ij} + i\varepsilon^{ij}\Gamma_{11})e_i^{\hat{a}}\Gamma_{\hat{a}}\nabla_j v, \end{aligned}\quad (3.35)$$

¹⁸The Wick rotation amounts to replacing the Minkowski signature metric with the Euclidean one, ε^{ij} with $-i\varepsilon^{ij}$ and taking into account that the fermions Θ become complex spinors, since there are no Majorana spinors in ten-dimensional Euclidean space. The complex conjugate spinors do not appear in the Wick rotated action so that the number of the fermionic degrees of freedom formally remains the same as before the Wick rotation. Note also that the Euclidean γ matrix is defined as $\gamma = i\Gamma^{0'}\Gamma^1$, where $\Gamma^{0'}$ is the Wick rotated Γ^0 . Thus $\gamma^2 = 1$ as in the case of Minkowski signature.

where $e_i^{\hat{a}}$ are the vielbeins of the instanton S^2 sphere parametrized by the worldsheet coordinates. To carry out the analysis of the fermionic modes in a covariant way, in (3.35) we have introduced the induced metric $h_{ij} = e_i^{\hat{a}} e_j^{\hat{b}} \delta_{\hat{a}\hat{b}}$ on the worldsheet instanton sphere S^2 . For a particular choice of the S^2 coordinates h_{ij} can be chosen to be conformally flat.

We see that ϑ and v decouple from each other and have the following equations of motion

$$h^{ij} e_i^{\hat{a}} \Gamma_{\hat{a}} (\nabla_j + \frac{1}{2} \mathcal{P}_8 \gamma \Gamma_{11} \Gamma_{\hat{a}} e_j^{\hat{a}}) (1 - \Gamma) \vartheta = 0, \quad (3.36)$$

$$h^{ij} e_i^{\hat{a}} \Gamma_{\hat{a}} \nabla_j (1 - \Gamma) v = 0, \quad (3.37)$$

where $\Gamma = -\frac{i}{2\sqrt{h}} \varepsilon^{ij} \Gamma_{ij} \Gamma_{11}$ coincides on the bosonic instanton configuration with the kappa-symmetry projector.

The equation (3.37) is the massless Dirac equation on S^2 which does not have non-trivial solutions, hence $(1 - \Gamma)v = 0$. The equation (3.36) is the ‘massive’ Dirac equation (with mass 1 in the inverse radius units) whose only regular solutions are the S^2 Killing spinors satisfying

$$(\nabla_j + \frac{1}{2} \gamma \Gamma_{11} \Gamma_{\hat{a}} e_j^{\hat{a}}) (1 - \Gamma) \vartheta = (\nabla_j + \frac{i}{2} \Gamma_{23} \gamma_7 \Gamma_{\hat{a}} e_j^{\hat{a}}) (1 - \Gamma) \vartheta = 0, \quad (3.38)$$

where $\gamma_7 = i\Gamma_{456789}$ and we have dropped the projector \mathcal{P}_8 since it commutes with $\Gamma_{\hat{a}}$ ($\hat{a} = 2, 3$) on S^2 and γ_7 . To see that (3.38) is indeed the Killing spinor equation on S^2 let us redefine the matrices $\Gamma_{\hat{a}}$ as follows

$$(\Gamma_{23} \Gamma_{\hat{a}}) \Rightarrow \Gamma_{\hat{a}} \quad (3.39)$$

and split the four-component spinor $(1 - \Gamma)\vartheta$ into the eigenvalues of γ_7 (recall that $(\gamma_7)^2 = 1$)

$$(1 - \Gamma)\vartheta = \vartheta_+ + \vartheta_-, \quad \vartheta_{\pm} = \frac{1}{2}(1 \pm \gamma_7)(1 - \Gamma)\vartheta. \quad (3.40)$$

Then (3.38) takes the form of the conventional Killing spinor equations on S^2 for two 2-component spinors ϑ_+ and ϑ_-

$$(\nabla_j \pm \frac{i}{2} \Gamma_{\hat{a}} e_j^{\hat{a}}) \vartheta_{\pm} = 0. \quad (3.41)$$

Thus we conclude that the string instanton wrapping S^2 has 4 fermionic zero modes associated with the S^2 Killing spinors ϑ_{\pm} .

T^2 instanton

When the string worldsheet wraps a T^2 (with coordinates y^i) in T^6 , i.e. $y^i = \xi^i$ the fermionic part (3.4) of the Wick rotated GS action reduces to

$$\mathcal{L}_{\text{GS}} = i\vartheta (1 - \Gamma) \Gamma^i \partial_i v + i v (1 - \Gamma) \Gamma^i \partial_i \vartheta + i v (1 - \Gamma) \Gamma^i \partial_i v - \frac{i}{2} v \gamma \gamma_7 (1 - \gamma_5) (1 - \Gamma) v, \quad (3.42)$$

where $\Gamma = -\frac{i}{2} \varepsilon^{ij} \Gamma_{ij} \Gamma_{11} = -\gamma_5 \gamma_{\bar{5}}$ is the kappa-symmetry projector in this background and $\gamma_{\bar{5}} = \Gamma_{3456}$ is the product of the gamma-matrices with the indices of the T^6 directions orthogonal to the instanton worldsheet.

The fermionic equations of motion which follow from (3.42) are

$$\mathcal{P}_{24} \Gamma^i \partial_i (1 - \Gamma) v - \frac{1}{2} \mathcal{P}_{24} \gamma \gamma_7 (1 - \gamma_5) (1 - \Gamma) v = -\mathcal{P}_{24} \Gamma^i \partial_i (1 - \Gamma) \vartheta, \quad (3.43)$$

or, equivalently,

$$\mathcal{P}_{24} \Gamma^i \partial_i (1 - \Gamma) v + \frac{1}{2} \mathcal{P}_{24} \gamma \Gamma_{11} (1 - \gamma_{\bar{5}}) (1 - \Gamma) v = -\mathcal{P}_{24} \Gamma^i \partial_i (1 - \Gamma) \vartheta, \quad (3.44)$$

$$\mathcal{P}_8 \Gamma^i \partial_i (1 - \Gamma) v = 0. \quad (3.45)$$

These two equations can be combined into the single one

$$\Gamma^i \partial_i (1 - \Gamma) v - \frac{1}{2} \gamma \gamma_7 (1 - \gamma_5) (1 - \Gamma) v = -\mathcal{P}_{24} \Gamma^i \partial_i (1 - \Gamma) \vartheta . \quad (3.46)$$

Hitting this equation with $\mathcal{P}_8 \Gamma^j \partial_j$ and taking into account (3.45) and the fact that $\mathcal{P}_8 \Gamma^j \mathcal{P}_8 = 0$ we find that ϑ should satisfy the ‘massless’ Laplace equation on T^2

$$\partial^i \partial_i (1 - \Gamma) \vartheta = 0 . \quad (3.47)$$

The zero modes of the Laplace operator on T^2 are constants, hence the T^2 string instanton has 4 fermionic constant modes $(1 - \Gamma) \vartheta$, and eq. (3.46) reduces to

$$\Gamma^i \partial_i (1 - \Gamma) v - \frac{1}{2} \gamma \gamma_7 (1 - \gamma_5) (1 - \Gamma) v = 0 . \quad (3.48)$$

From this equation we read that the eight¹⁹ modes $\frac{1}{2}(1 + \gamma_5)(1 - \Gamma)v$ should satisfy the massless Dirac equation

$$\Gamma^i \partial_i (1 - \Gamma) (1 + \gamma_5) v = 0 , \quad (3.49)$$

and hence are constants. We are thus left with

$$\Gamma^i \partial_i (1 - \Gamma) v_- - \gamma \gamma_7 (1 - \Gamma) v_- = 0 , \quad v_- = \frac{1}{2} (1 - \gamma_5) v , \quad (3.50)$$

$$\mathcal{P}_8 \Gamma^i \partial_i (1 - \Gamma) v_- = 0 , \quad (3.51)$$

where the last equation is the consequence of (3.45) and (3.49).

To analyze eq. (3.51), let us use the explicit form of the projectors \mathcal{P}_8 and \mathcal{P}_{24} which is similar to that in the case of the GS string instanton on CP^3 (see eq. (4.42) of [37])

$$\mathcal{P}_8 = \frac{1}{8} (2 + J) = \frac{1}{4} (1 + \rho^3 \tilde{J}) (1 - \gamma_5) , \quad \mathcal{P}_{24} = \frac{1}{8} (6 - J) = \frac{1}{4} (3 + \gamma_5 - \rho^3 \tilde{J} (1 - \gamma_5)) , \quad (3.52)$$

where (the indices 3, 4, 5, 6 denoting the directions in T^6 orthogonal to the instanton worldsheet)

$$\rho_3 = -\frac{i}{2} \varepsilon^{ij} \Gamma_{ij} , \quad \gamma_5 = \Gamma_{3456} , \quad \tilde{J} = -\frac{i}{4} J_{\tilde{a}\tilde{b}} \gamma^{\tilde{a}\tilde{b}} = -\frac{i}{8} J_{\tilde{a}\tilde{b}} \gamma^{\tilde{a}\tilde{b}} (1 - \gamma_5) , \quad \tilde{J}^2 = \frac{1}{2} (1 - \gamma_5) \quad (3.53)$$

Inserting the explicit form (3.52) of \mathcal{P}_8 into (3.51) we have

$$\begin{aligned} \mathcal{P}_8 \Gamma^i \partial_i (1 - \Gamma) v_- &= \frac{1}{4} \Gamma^i \partial_i (1 - \rho^3 \tilde{J}) (1 - \gamma_5) (1 - \Gamma) v_- \\ &= \Gamma^i \partial_i \mathcal{P}_{24} (1 - \Gamma) v_- - \frac{1}{2} \Gamma^i \partial_i (1 + \gamma_5) (1 - \Gamma) v_- = \Gamma^i \partial_i \mathcal{P}_{24} (1 - \Gamma) v_- = 0 , \end{aligned} \quad (3.54)$$

where the term $(1 + \gamma_5) (1 - \Gamma) v_-$ is zero since $\Gamma = -\gamma_5 \gamma_5$ and $(1 + \gamma_5) (1 - \Gamma) (1 - \gamma_5) = (1 + \gamma_5) (1 + \gamma_5 \gamma_5) (1 - \gamma_5) = 0$. From eq. (3.54) it follows that $(1 - \Gamma) v_-$ is constant and from (3.50) it follows that it is actually zero.

We thus conclude that the T^2 string instanton has 12 constant fermionic zero modes: 8 modes $(1 + \gamma_5)(1 - \Gamma)v$ and 4 modes $(1 - \Gamma)\vartheta$.

3.4.2 Hybrid model action

Let us now discuss the corresponding fermionic term in the hybrid model action and compare it with the GS action.

¹⁹This number follows from the analysis of [37]. See Subsection 4.2 therein.

S^2 instanton

On the S^2 instanton background the fermionic part of the hybrid model Lagrangian takes the following form

$$\mathcal{L}_{\text{ferm}} = 2i \nabla_i \vartheta \gamma \Gamma_{11} (\sqrt{-h} h^{ij} + i \varepsilon^{ij} \Gamma_{11}) (\nabla_j \vartheta + \frac{1}{2} \gamma \Gamma_{11} \Gamma_{\hat{b}} e_j^{\hat{b}} \vartheta) + \frac{i}{2} \bar{\psi}^{a'} \gamma_{\hat{a}} e^{\hat{a}i} \nabla_i \psi^{a'}, \quad (3.55)$$

where $e_i^{\hat{a}}(\xi)$ ($\hat{a} = 2, 3$) are the vielbeins of the instanton sphere, $h_{ij} = e_i^{\hat{a}} e_{j\hat{a}}$, $\gamma^{\hat{a}}$ are the worldsheet gamma-matrices and $\psi^{a'}(\xi)$ ($a' = 4, 5, 6, 7, 8, 9$) are 6 two-component worldsheet RNS-like fermions on T^6 .

The equations of motion of ψ are massless Dirac equations on S^2

$$\nabla \psi = 0, \quad \nabla = e^{\hat{a}i} \gamma_{\hat{a}} \nabla_i \quad (3.56)$$

which do not have non-trivial regular solutions. The equations of motion of ϑ are

$$\frac{1}{\sqrt{h}} \nabla_i (\sqrt{h} h^{ij} + i \varepsilon^{ij} \Gamma_{11}) (\nabla_j \vartheta + \frac{1}{2} \gamma \Gamma_{11} \Gamma_{\hat{b}} e_j^{\hat{b}} \vartheta) = 0. \quad (3.57)$$

We can rewrite them as

$$\begin{aligned} & (\nabla_i \nabla^i + \frac{i}{2} \Gamma_{23} \Gamma_{11}) \vartheta + \frac{1}{2} \gamma \Gamma_{11} \nabla \vartheta + \frac{i}{2\sqrt{h}} \gamma \Gamma_{11} \Gamma_{11} \varepsilon^{ij} \Gamma_j \nabla_i \vartheta \\ & = (\nabla_i \nabla^i + \frac{i}{2} \Gamma_{23} \Gamma_{11}) \vartheta + \frac{1}{2} \gamma \Gamma_{11} (1 + \Gamma) \nabla \vartheta = 0 \end{aligned} \quad (3.58)$$

To arrive at (3.58) we have used that on S^2 of unit radius

$$\frac{1}{\sqrt{h}} \varepsilon^{ij} \nabla_i \nabla_j = \frac{1}{2} \Gamma_{23}, \quad \varepsilon^{ij} \Gamma_j = -\frac{1}{2} \varepsilon^{jk} \Gamma_{jk} \Gamma_i = -\Gamma_{23} \Gamma_i. \quad (3.59)$$

$\Gamma = -i \Gamma_{23} \Gamma_{11}$ is the same as in eq. (3.37) and $\gamma = i \Gamma^{01}$ (with $\gamma^2 = 1$) is the product of Wick rotated AdS_2 Dirac matrices.

To analyze the solutions of (3.58), let us multiply it by the projectors $\frac{1}{2}(1 \pm \Gamma)$. This gives the equations of motion for $\vartheta_{\pm} = \frac{1}{2}(1 \pm \Gamma) \vartheta$

$$(\nabla_i \nabla^i - \frac{1}{2}) \vartheta_+ + \gamma \Gamma_{11} \nabla \vartheta_+ = 0, \quad (3.60)$$

$$(\nabla_i \nabla^i + \frac{1}{2}) \vartheta_- = 0. \quad (3.61)$$

Using (3.59) we may get from (3.60)

$$\nabla (\nabla - \gamma \Gamma_{11}) \vartheta_+ = 0. \quad (3.62)$$

Note that the fermionic operator factorizes into the massless Dirac operator times the massive GS operator (cf. (3.36), (3.41)). In the case of the sphere, the only non-trivial solutions of (3.62) are the S^2 Killing spinors satisfying

$$(\nabla - \gamma \Gamma_{11}) \vartheta_+ = 0 \quad (3.63)$$

as in (3.36), (3.41). This gives us 4 fermionic zero modes as in the case of the GS instanton on S^2 . Note that from eqs. (3.60) and (3.63) it follows that $\vartheta_+ = \frac{1}{2}(1 + \Gamma) \vartheta$ are eigenfunctions of the Laplace operator with the eigenvalue $-\frac{1}{2}$ ($\nabla_i \nabla^i + \frac{1}{2}) \vartheta_+ = 0$. According to (3.61), four ϑ_- are the eigenfunctions of the Laplace operator with the eigenvalue $-\frac{1}{2}$. Hence, they can also be associated with the S^2 Killing spinors with an effective mass ± 1 .

Thus we seem to get an additional $8 = 4 \times 2$ fermionic zero modes in the hybrid model as compared to the corresponding GS case. However, the comparison of the total expressions for the two partition functions, that we do not attempt here, requires a proper definition of the path integral measure, as was already emphasized on the example of the infinite string in AdS_2 in section 2.4.1. In fact, since this case is essentially an analytic continuation of that discussion we expect a match also in this case.

T^2 instanton

In this case the fermionic part of the hybrid model Lagrangian takes the following form

$$\mathcal{L}_{\text{ferm}} = 2i \partial^i \vartheta \gamma \Gamma_{11} \partial_j \vartheta + \frac{i}{2} \bar{\psi}^{a'} \gamma^i \partial_i \psi^{a'} . \quad (3.64)$$

From eq.(3.64) we see that eight ϑ should obey the Laplace equation on T^2 and twelve $\psi^{a'}$ should satisfy the massless Dirac equation, i.e.

$$\partial^i \partial_i \vartheta = 0, \quad \gamma^i \partial_i \psi^{a'} = 0 . \quad (3.65)$$

These equations have only constant solutions on T^2 , so that we get 20 zero modes associated with them. Though this number does not match the one of the GS string instanton on T^2 which has 12 constant fermionic modes, the difference should presumably be accounted for by the contribution of the ghost sector of the hybrid model.

4 Concluding remarks

In this paper we have demonstrated the semiclassical equivalence between the Green–Schwarz and the pure–spinor formulations of the $AdS_5 \times S^5$ superstring expanded near generic classical string solutions, extending earlier work in this direction [27, 28]. We have also studied a similar relation between the $AdS_2 \times S^2 \times T^6$ GS superstring and the corresponding hybrid model.

We have shown that in the $AdS_5 \times S^5$ pure–spinor model expanded around a classical string solution, half of the fermionic modes enter the quadratic fluctuation action only linearly. Therefore, they can be integrated out, contributing a massless $2d$ d’Alembertian operator determinant to the one-loop partition function. The action for the remaining half of the fermions takes a form which resembles the structure of the GS superstring action with the first–derivative terms being the same but with the GS “mass” term being replaced by a second–derivative fermionic kinetic term. We have found that the contribution of these remaining fermions to the one-loop pure–spinor partition function is the same as the non-trivial “massive” determinant of the GS fermions times an additional massless determinant, with the latter being cancelled against the ghost sector contributions.

One may expect a similar equivalence between the GS and the pure-spinor formulation also for the case of the $AdS_2 \times S^2 \times T^6$ background. In this case, however, there exists also a simpler version of a “pure–spinor”-like formulation – the hybrid model of [5] in which the $AdS_2 \times S^2$ and T^6 parts are essentially decoupled in the action (in contrast to what happens in the GS case [11]). We have found that there is a similar semiclassical correspondence between the $AdS_2 \times S^2 \times T^6$ GS theory and the hybrid model by considering particular cases when the classical strings move only in $AdS_2 \times S^2$ and two simple cases when the string also moves in or wraps around a circle in T^6 . We also compared the spectra of the fermionic zero modes of S^2 and T^2 worldsheet instantons in the GS and the hybrid formulations. It would be interesting to complete this analysis in order to prove the semiclassical equivalence between these two superstring formulations for a generic motion of the string in $AdS_2 \times S^2 \times T^6$.

Another open problem is to repeat a similar study in the case of superstrings in $AdS_3 \times S^3 \times T^4$. There exists three a priori different formulations of superstring theory in this background (which can be supported, in general, by a combination of R-R and NS-NS 3-form fluxes): (i) The Green–Schwarz formulation based on the $PSU(1,1|2) \times PSU(1,1|2)/SU(1,1) \times SU(2)$ supercoset sigma model with a particular WZ term, enlarged with 4 bosonic T^4 degrees of freedom and additional 16 fermionic ones (see [39, 40, 10, 41, 42] and references there);²⁰ (ii) the supercoset hybrid model with a “pure–spinor”-like (second-derivative fermion) sigma model action [5, 13] (for the same supercoset with the same WZ term and the same number of 16 supersymmetries as in the GS

²⁰In the absence of R-R flux there is of course also the standard RNS formulation based on $2d$ supersymmetric $SU(1,1) \times SU(2)$ WZW model.

case); (iii) the supergroup hybrid model [12] based on the $PSU(1, 1|2)$ sigma model (with an extra WZ term in the case of non-zero NS-NS 3-form flux) in which only 8 out of 16 supersymmetries are manifest.²¹ It would be interesting to check that these three formulations are indeed equivalent in the semiclassical expansion.

Acknowledgements

The authors are grateful to I. Bandos, N. Berkovits, J. Gomis, R. Roiban, I. Samsonov, M. Tonin, B. Vallilo and K. Zarembo for useful discussions and comments. A.A.T. is grateful to R. Roiban for a collaboration on a related unpublished work [27]. Work of A.C., D.S. and A.A.T. was partially supported by the Uni-PD Research Grant CPDA119349. D.S. was also supported in part by the MIUR-PRIN contract 2009-KHZKRX. A.C. is thankful to Angelo della Riccia Foundation for the grant to visit Nordita. The work of A.A.T. was supported by the STFC grant ST/J000353/1 and by the ERC Advanced grant No.290456. The research of L.W. was supported in part by NSF grants PHY-0555575 and PHY-0906222.

Appendix A. Notation and conventions

The flat metric is $\eta_{AB} = (-, + \dots, +)$. The $D = 10$ gamma matrices Γ^A and Γ^{11} are real and

$$\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}.$$

Contracted with the complex charge conjugation matrix C the matrices

$$C\Gamma^{\hat{A}}, \quad C\Gamma^{\hat{A}\hat{B}}, \quad C\Gamma^{\hat{A}_1 \dots \hat{A}_5} \quad \text{are symmetric} \quad (\hat{A}, \hat{B}, \dots = 0, \dots, 9, 11)$$

and the matrices

$$C, \quad C\Gamma^{\hat{A}\hat{B}\hat{C}}, \quad C\Gamma^{\hat{A}\hat{B}\hat{C}\hat{D}} \quad \text{are anti-symmetric.}$$

Any two spinors are contracted as follows

$$\theta^\alpha C_{\alpha\beta} \psi^\beta \equiv \theta\psi, \quad \theta C\Gamma^{\hat{A}\dots} \psi = \theta^\alpha (C\Gamma^{\hat{A}\dots})_{\alpha\beta} \psi^\alpha \equiv \theta\Gamma^{\hat{A}\dots} \psi.$$

Note that to simplify the notation we do not indicate explicitly the charge conjugation matrix C in bilinear combinations of the spinors contracted with $C\Gamma^{\hat{A}\dots}$.

Since the matrices $C\Gamma^{\hat{A}}$ and $C\Gamma^{\hat{A}\hat{B}}$ are symmetric, we have

$$(\Gamma^{\hat{A}}\vartheta)C\psi = -\vartheta C\Gamma^{\hat{A}}\psi \equiv -\vartheta\Gamma^{\hat{A}}\psi, \quad (\Gamma^{\hat{A}\hat{B}}\vartheta)C\psi = -\vartheta C\Gamma^{\hat{A}\hat{B}}\psi \equiv -\vartheta\Gamma^{\hat{A}\hat{B}}\psi.$$

Appendix B. Folded spinning string in $R \times S^2 \subset AdS_2 \times S^2 \times T^6$

To illustrate the semiclassical equivalence of the GS formulation and the hybrid model for $AdS_2 \times S^2 \times T^6$ that we argued for in general when the string is moving entirely in $AdS_2 \times S^2$ let us consider the case of a folded spinning string moving in S^2 . The corresponding solution in conformal gauge is

$$t = \kappa\tau, \quad \theta = \theta(\sigma), \quad \varphi = w\tau, \quad (B.1)$$

$$\sin\theta = \sqrt{q} \operatorname{sn}(w\sigma|q), \quad q = \sin^2\theta_0 = \frac{\kappa^2}{w^2}, \quad w = \frac{2}{\pi}K(q), \quad (B.2)$$

where t is the AdS_2 time coordinate, θ and φ are the spherical coordinates of S^2 , and K is the elliptic integral. Here $\theta'^2 = \kappa^2 - w^2 \sin^2\theta$ so that the induced metric is

$$ds^2 = \theta'^2(-d\tau^2 + d\sigma^2). \quad (B.3)$$

²¹For some recent discussions of this supergroup hybrid model see, e.g., [43, 44, 45] and references there.

The non-zero worldsheet pullbacks of the $AdS_2 \times S^2 \times T^6$ vielbeins are

$$e_\tau^0 = \kappa, \quad e_\sigma^2 = \theta', \quad e_\tau^3 = \omega \sin \theta \quad (B.4)$$

and the covariant derivative acting on the fermions is

$$\nabla_i = \partial_i - \frac{1}{4} \omega_i^{ab} \Gamma_{ab} = (\partial_\tau - \frac{1}{2} w \cos \theta \Gamma_{23}, \partial_\sigma), \quad \nabla_\pm = \partial_\pm - \frac{1}{2} w \cos \theta \Gamma_{23}. \quad (B.5)$$

In the conformal gauge, the bosonic fluctuations around this solution are described by the following Lagrangian (see, e.g., [46])

$$L_{\text{boson}} = -\frac{1}{2}(\partial \tilde{t})^2 + \frac{1}{2}[(\partial \tilde{\rho})^2 + k^2 \tilde{\rho}^2] + \frac{1}{2}(\partial \tilde{\theta})^2 + \frac{1}{2} \sin^2 \theta (\partial \tilde{\varphi})^2 - 2w \cos \theta \sin \theta \tilde{\theta} \partial_\tau \tilde{\varphi} - \frac{1}{2}(1 - 2 \sin^2 \theta) w^2 \tilde{\theta}^2 + \frac{1}{2}(\partial \tilde{y})^2. \quad (B.6)$$

Defining $\eta = \sin \theta \tilde{\varphi}$ one gets

$$L_{\text{boson}} = -\frac{1}{2}(\partial \tilde{t})^2 + \frac{1}{2}[(\partial \tilde{\rho})^2 + k^2 \tilde{\rho}^2] + \frac{1}{2}[(\nabla \tilde{\theta})^2 + w^2 \sin^2 \theta \tilde{\theta}^2] + \frac{1}{2}[(\nabla \eta)^2 - \theta'^2 \eta^2] + \frac{1}{2}(\partial \tilde{y})^2, \quad (B.7)$$

where $\nabla \eta = (\partial_\tau \eta - w \cos \theta \tilde{\theta}, \partial_\sigma \eta)$ and $\nabla \tilde{\theta} = (\partial_\tau \tilde{\theta} + w \cos \theta \eta, \partial_\sigma \tilde{\theta})$. Thus we have three “massive” bosonic modes, one in AdS_2 and two in S^2 , with the sum of squares of their masses being

$$\sum m_b^2 = 2 w^2 \sin^2 \theta. \quad (B.8)$$

Green–Schwarz action

The quadratic fermionic term in the GS action (3.3) in this background takes the following form

$$\mathcal{L}_{\text{GS}} = -i \vartheta^1 \not\! \partial_+ \nabla_- \vartheta^1 - i \vartheta^3 \not\! \partial_- \nabla_+ \vartheta^3 + i \vartheta^1 \not\! \partial_+ \Gamma_{01} \not\! \partial_- \vartheta^3 - i v^1 \not\! \partial_+ \nabla_- v^1 - i v^3 \not\! \partial_- \nabla_+ v^3. \quad (B.9)$$

Making use of the explicit form of the spin connection (B.5), let us perform the following redefinition of $\vartheta^{1,3}$ and $v^{1,3}$

$$\vartheta^1 = \frac{1}{\sqrt{\kappa}} e^{\frac{f(\sigma)}{2} \Gamma_{23}} \tilde{\vartheta}^1, \quad \vartheta^3 = \frac{1}{\sqrt{\kappa}} e^{-\frac{f(\sigma)}{2} \Gamma_{23}} \tilde{\vartheta}^3, \quad v^1 = \frac{1}{\sqrt{\kappa}} e^{\frac{f(\sigma)}{2} \Gamma_{23}} \tilde{v}^1, \quad v^3 = \frac{1}{\sqrt{\kappa}} e^{-\frac{f(\sigma)}{2} \Gamma_{23}} \tilde{v}^3, \quad (B.10)$$

where

$$f = \arcsin \frac{\omega \sin \theta}{\kappa}, \quad \partial_\sigma f = \omega \cos \theta, \quad \cos f = \frac{\theta'}{\kappa} = \frac{\sqrt{k^2 - \omega^2 \sin^2 \theta}}{\kappa}. \quad (B.11)$$

Then, taking into account that

$$e^{-\frac{f}{2} \Gamma_{23}} \not\! \partial_- e^{\frac{f}{2} \Gamma_{23}} = \kappa (\Gamma_0 - \Gamma_2) = \kappa \Gamma_-, \quad e^{\frac{f}{2} \Gamma_{23}} \not\! \partial_+ e^{-\frac{f}{2} \Gamma_{23}} = \kappa (\Gamma_0 + \Gamma_2) = \kappa \Gamma_+,$$

we get

$$\mathcal{L}_{\text{GS}} = -2i \tilde{\vartheta}_-^1 \Gamma_0 \partial_- \tilde{\vartheta}_-^1 - 2i \tilde{\vartheta}_+^3 \Gamma_0 \partial_+ \tilde{\vartheta}_+^3 - 4i \omega \sin \theta \tilde{\vartheta}_-^1 \Gamma_{13} \tilde{\vartheta}_+^3 + 2i \tilde{v} \not\! \partial \tilde{v}, \quad (B.12)$$

where $\tilde{\vartheta}_-^1 = \frac{1}{2}(1 - \Gamma_{02}) \tilde{\vartheta}^1$ and $\tilde{\vartheta}_+^3 = \frac{1}{2}(1 + \Gamma_{02}) \tilde{\vartheta}^3$. We thus find that two pairs of fermions $\tilde{\vartheta}$ have the “mass” $m = \omega \sin \theta$ and 6 pairs of the fermions \tilde{v} are massless.

Hybrid model action

Upon performing the redefinition (B.10) of $\vartheta^{1,3}$ and splitting them into $\tilde{\vartheta}_{\pm}^{1,3} = \frac{1}{2}(1 \pm \Gamma_{02})\tilde{\vartheta}^{1,3}$, the fermionic part of the hybrid model Lagrangian (3.9) takes the form

$$\begin{aligned} \mathcal{L}_H = & \frac{4i \cos f}{\kappa} \partial_+ \tilde{\vartheta}_-^1 \Gamma_{12} \partial_- \tilde{\vartheta}_-^3 - \frac{4i \cos f}{\kappa} \partial_+ \tilde{\vartheta}_+^1 \Gamma_{12} \partial_- \tilde{\vartheta}_+^3 - \frac{4i \sin f}{\kappa} \partial_+ \tilde{\vartheta}_-^1 \Gamma_{13} \partial_- \tilde{\vartheta}_+^3 \\ & + \frac{4i \sin f}{\kappa} \partial_+ \tilde{\vartheta}_+^1 \Gamma_{13} \partial_- \tilde{\vartheta}_-^3 + 2i \tilde{\vartheta}_+^1 \Gamma_0 \partial_+ \tilde{\vartheta}_+^1 + 2i \tilde{\vartheta}_-^3 \Gamma_0 \partial_- \tilde{\vartheta}_-^3. \end{aligned} \quad (\text{B.13})$$

Using that $\sin f = \frac{\omega \sin \theta}{\kappa}$ this Lagrangian can be written as

$$\begin{aligned} \mathcal{L}_H = & -\frac{4i\omega \sin \theta}{\kappa^2} (\partial_+ \tilde{\vartheta}_-^1 - \cot f \Gamma_{23} \partial_+ \tilde{\vartheta}_+^1) \Gamma_{13} (\partial_- \tilde{\vartheta}_+^3 + \cot f \Gamma_{23} \partial_- \tilde{\vartheta}_-^3) \\ & + \frac{4i}{\omega \sin \theta} \partial_+ \tilde{\vartheta}_+^1 \Gamma_{13} \partial_- \tilde{\vartheta}_-^3 + 2i \tilde{\vartheta}_+^1 \Gamma_0 \partial_+ \tilde{\vartheta}_+^1 + 2i \tilde{\vartheta}_-^3 \Gamma_0 \partial_- \tilde{\vartheta}_-^3. \end{aligned} \quad (\text{B.14})$$

Note that in the first line of (B.14) the pairs of terms in each bracket have the same chirality. This is important for performing the integration of $\tilde{\vartheta}_-^1$ and $\tilde{\vartheta}_+^3$ which enter the Lagrangian linearly. Under an appropriate assumption about the path integral measure (cf. [29]) they contribute to the partition function only with a massless determinant factor $(\det \partial_+ \partial_-)^4$.

After the modes $\tilde{\vartheta}_-^1$ and $\tilde{\vartheta}_+^3$ are integrated out we are left with the last line of (B.14), which is a counterpart of the GS Lagrangian (B.12). By the same reasoning as used for a generic classical solution in section 3 one can verify that this part of the Lagrangian (B.14) indeed describes the same number (two) of massive fermions with the same mass as in the GS string case plus two massless fermions.

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